

EXISTENCE AND UNIQUENESS FOR MEAN FIELD EQUATIONS ON MULTIPLY CONNECTED DOMAINS AT THE CRITICAL PARAMETER

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ABSTRACT. We consider the mean field equation:

$$(1) \quad \begin{cases} \Delta u + \rho \frac{e^u}{\int_{\Omega} e^u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is an open and bounded domain of class C^1 . In his 1992 paper, Suzuki proved that if Ω is a simply-connected domain, then equation (1) admits a unique solution for $\rho \in [0, 8\pi)$. This result for Ω a simply-connected domain has been extended to the case $\rho = 8\pi$ by Chang, Chen and the second author. However, the uniqueness result for Ω a multiply-connected domain has remained a long standing open problem which we solve positively here for $\rho \in [0, 8\pi]$. To obtain this result we need a new version of the classical Bol's inequality suitable to be applied on multiply-connected domains.

Our second main concern is the existence of solutions for (1) when $\rho = 8\pi$. We obtain necessary and sufficient condition for the solvability of the mean field equation at $\rho = 8\pi$ which is expressed in terms of the Robin's function γ for Ω . For example, if equation (1) has no solution at $\rho = 8\pi$, then γ has a unique nondegenerate maximum point. As a by product of our results we solve the long-standing open problem of the equivalence of canonical and microcanonical ensembles in the Onsager's statistical description of two-dimensional turbulence on multiply-connected domains.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 and $H_0^1(\Omega)$ denote the standard Sobolev space of functions with vanishing boundary

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values. We define the functional $J_\rho : H_0^1(\Omega) \mapsto \mathbb{R}$ as

$$J_\rho(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \rho \log \int_\Omega e^u, \quad \forall u \in H_0^1(\Omega).$$

The Euler-Lagrange equation for J_ρ has the following form:

$$(1.1) \quad \begin{cases} \Delta u + \rho \frac{e^u}{\int_\Omega e^u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operation in \mathbb{R}^2 . Problem (1.1) is relevant to many research areas in mathematics and it has been extensively studied for the past three decades. In geometry, the equation in (1.1) is strictly related with the local version of the prescribed constant Gaussian curvature problem on two dimensional surfaces see for example [2], [4], [18], [20], [30]. In statistical mechanics, problem (1.1) is the mean field limit of the Gibbs measures associated with the Onsager's description of turbulent Euler flows, as studied by Caglioti, Lions, Marchioro and Pulvirenti [10], [11], Kiessling [24], Chanillo and Kiessling [15] and Lin [28]. Recently, it has attracted a lot of attention because it also appeared as a limiting equation in the self-dual Chern-Simons-Higgs model in a relativistic version of superconductivity and other gauge field theories, see [5], [9], [12], [14], [32], [42], [43], [44], [45], [47], [48] and references therein.

The classical Moser-Trudinger [38] inequality implies that if $\rho \leq 8\pi$, then J_ρ is bounded from below and coercive on $H_0^1(\Omega)$. In this situation it is not hard to find a global minimizer of J_ρ in $H_0^1(\Omega)$. Thus, problem (1.1) always admits at least one solution for $\rho < 8\pi$. In general, we can compute a degree counting formula for problem (1.1) whenever $\rho \neq 8\pi m$, where m is a positive integer. C.C. Chen and the second author [17], [18] proved that if Ω is not simply-connected, then the Leray-Schauder degree corresponding to the resolvent operator naturally associated with (1.1) does not vanish for any $\rho \neq 8\pi m$. Therefore (1.1) admits at least one solution for $\rho \neq 8\pi m$ and Ω not simply-connected. See also [37] for another derivation of the Leray-Schauder degree for (1.1) on closed surfaces.

In case $\rho \neq 8\pi m$, then the degree formulas obtained in [18] do depend only on the topology of Ω . On the contrary, if $\rho = 8\pi m$, then the existence of solutions for (1.1) will depend also on the geometry of Ω . For example if Ω is a ball, then (1.1) has no solutions for $\rho = 8\pi$, while if Ω is a long and thin ellipse and/or rectangle (see [11] and in particular [13]), then (1.1) admits at least one solution for $\rho = 8\pi$. Thus, it is natural to ask the following question:

Q: What kind of geometries do allow the existence of a solution for (1.1) with $\rho = 8\pi$?

In case Ω is simply-connected, Chang, Chen and the second author [13] already gave an answer to this question. To state their result, we should first recall the definition of the Robin's function for Ω . We let $G(x, p)$ denote the Green's function of $-\Delta$ with Dirichlet boundary conditions, uniquely

defined by

$$\begin{cases} -\Delta G(x, p) &= \delta_p & \text{in } \Omega, \\ G(x, p) &= 0 & \text{on } \partial\Omega, \end{cases}$$

and set

$$(1.2) \quad \begin{cases} \tilde{G}(x, p) = G(x, p) + \frac{1}{2\pi} \log |x - p|, \\ \gamma(p) = \tilde{G}(p, p). \end{cases}$$

Hence γ denotes the Robin's function relative to Ω and satisfies

$$\lim_{p \rightarrow \partial\Omega} \gamma(p) = -\infty.$$

Let q be a critical point of $\gamma(p)$. Clearly q is also a critical point of $\tilde{G}(x, q)$ with respect to the x variable, i.e.

$$(1.3) \quad \nabla_x \tilde{G}(x, q) = 0 \text{ at } x = q.$$

Let $D(q)$ be defined by

$$(1.4) \quad D(q) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B(q, \varepsilon)} \frac{e^{8\pi(\tilde{G}(x, q) - \gamma(q))} - 1}{|x - q|^4} - \int_{\Omega^c} \frac{dx}{|x - q|^4},$$

where $B(q, r)$ denotes the ball of center q and radius r . Note that in a neighborhood of q ,

$$(1.5) \quad e^{8\pi(\tilde{G}(x, q) - \gamma(q))} - 1 = \sum a_{ij}(x_i - q_i)(x_j - q_j) + O(|x - q|^3),$$

where, since $\tilde{G}(x, q)$ is harmonic in Ω , $a_{11} + a_{22} = 0$. In particular, by using (1.5), one can check that the limit in (1.4) always exists.

Now we can state the main theorem in [13].

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and simply-connected domain of class C^1 . Then (1.1) admits at least one solution for $\rho = 8\pi$ if and only if there exists a maximum point q of γ such that $D(q) > 0$.*

As an application of Theorem A, consider a dumbbell domain Ω_δ with two disjoint balls B_1, B_2 connected by a tube of small width $\delta > 0$. Let r_1, r_2 be the radius of B_1 and B_2 . If $r_2 \neq r_1$, then, by using Theorem A, we can prove that (1.1) with $\Omega \equiv \Omega_\delta$ has no solutions for $\rho = 8\pi$ provided that δ is sufficiently small. However if $r_1 = r_2$ and Ω_δ is further assumed to be symmetric with respect to y -axis, then (1.1) admits a solution for $\rho = 8\pi$ for any δ sufficiently small. See [13] for a proof of these facts and further examples.

One of our aims is to extend Theorem A to any bounded domain of class C^1 . In fact we have

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 . Then (1.1) admits at least one solution for $\rho = 8\pi$ if and only if there exists a maximum point q of γ such that $D(q) > 0$.*

An interesting application of Theorem 1.1 is the case $\Omega = \Omega_\varepsilon = B(0, 1) \setminus B(x_0, \varepsilon)$. If $x_0 = 0$, then (1.1) admits a solution (see for example [11]) for $\rho = 8\pi$ on Ω_ε for any $\varepsilon \in (0, 1)$. However, if $x_0 \neq 0$, then the Robin's function for Ω_ε (which we denote here by $\gamma_\varepsilon(x)$) converges to the Robin's function for $B(0, 1)$ (which we denote here by $\gamma(x)$) on any compact subset of $B(0, 1) \setminus \{x_0\}$, see for example [3] p.198-199. Since 0 is the only maximum point of γ then any maximum point q_ε of γ_ε must converge to 0 as $\varepsilon \rightarrow 0$. But the quantity $D(0)$ relative to $B(0, 1)$ is equal to -1 so that $D_\varepsilon(q_\varepsilon) < 0$ provided that ε is small. Hence Theorem 1.1 implies that (1.1) on $\Omega = \Omega_\varepsilon$ has no solutions at all for $\rho = 8\pi$ and ε small enough.

Consider the set of those bounded domains of class C^1 such that (1.1) has no solutions for $\rho = 8\pi$. An interesting consequence of Theorem 1.1 is the closeness of this set of domains under C^1 deformations. This property is essentially due to the fact that the quantity D in (1.3) is stable under C^1 deformations.

Corollary 1.1. *Let $\{\Omega_n\} \subset \mathbb{R}^2$ be a sequence of open and bounded domains of class C^1 and suppose that Ω_n converges to Ω in C^1 as $n \rightarrow +\infty$, where Ω is an open and bounded domain of class C^1 too. Suppose that equation (1.1) on Ω_n has no solutions for $\rho = 8\pi$. Then equation (1.1) on Ω has no solutions for $\rho = 8\pi$ as well.*

Another interesting consequence of Theorem 1.1 is the deep connection between the sign of $D(q)$ at a maximum point q of γ , the solvability of equation (1.1) for $\rho = 8\pi$ and the geometry of Ω . As a consequence of (1.5) we see that $D(q)$ is well-defined whenever q is a critical point of γ . We will show in section 4 that the following holds:

Corollary 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 . If $D(p) \leq 0$ for a critical point p of γ , then p must be a maximum point. In particular it is the unique maximum point and is nondegenerate.*

It is rather interesting to note that the Robin function is an elementary function of two variables which carries some geometric information about Ω [3], but which has no apparent direct connections with (1.1). Nevertheless, we are not aware of any "elementary" proof of the result in Corollary 1.2 which make no use of (1.1).

Remark 1.1. *As a straightforward consequence of Theorem 1.1 and Corollary 1.2 we obtain a result anticipated in the abstract, that is, if no solutions exist for (1.1) with $\rho = 8\pi$ then the Robin function γ for Ω admits a unique and nondegenerate maximum point.*

In other words, if γ has more than one maximum point, then the quantity D , evaluated at any critical point of γ , must be positive. Thus, we have another consequence of Theorem 1.1.

Corollary 1.3. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 . Suppose that γ , the Robin's function for Ω , has more than one maximum point. Then (1.1) admits at least one solution at $\rho = 8\pi$.*

Finally, the following criterion turns out to be very useful [13], [7] to prove existence/non-existence of a solution for $\rho = 8\pi$. Let us define

$$\mathcal{I}_{8\pi}(\Omega) := \inf_{u \in H_0^1(\Omega)} J_\rho(u).$$

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and multiply-connected domain of class C^1 . Then*

$$(1.6) \quad \frac{1}{8\pi} \mathcal{I}_{8\pi}(\Omega) \leq -1 - \log(\pi) - 4\pi \sup_{x \in \overline{\Omega}} \gamma(x),$$

and (1.1) admits at least one solution at $\rho = 8\pi$ if and only if the strict inequality holds.

Remark 1.2. *Corollary 1.4 is false in general if we consider the analogue version of (1.1) on the flat two-torus with periodic boundary conditions. In fact, it has been shown in [35], [36] that if the Green's function $G(\tau; 0)$ for the torus has five critical points, then the equality holds in (1.6) and the analogue version of (1.1) has one solution for $\rho = 8\pi$, which is a counterexample to Corollary 1.4 when the domain is a torus.*

We observe that both Theorem 1.1 and Corollary 1.3 state the existence of at least one solution. We can say much more concerning this point. Indeed, it turns out that the proof of Theorem 1.1 heavily relies on the fact that (1.1) admits at most one solution for $\rho \in [0, 8\pi]$. In case Ω is simply-connected, the uniqueness of solutions for (1.1) with $\rho \in [0, 8\pi]$ has been proved by Suzuki in [46]. That result has been improved by Chang, Chen and the second author [13] for $\rho = 8\pi$ and then generalized by the authors in [7] to cover the case where Dirac data are included in (1.1). However, the uniqueness for (1.1) on multiply-connected domains has remained an open problem for a long time. In this paper, we answer this question affirmatively.

Theorem 1.2. *Let Ω be an open, bounded and multiply-connected domain of class C^1 . Then equation (1.1) admits at most one solution for $\rho \leq 8\pi$. Moreover, the first eigenvalue of the corresponding linearized problem is strictly positive for any $\rho \leq 8\pi$.*

The proof of Theorem 1.2 relies on our third new result which is the Bol's inequality on multiply-connected domains. Let $w \in C^2(\Omega)$ satisfy the differential inequality:

$$(1.7) \quad \Delta w + e^w \geq 0 \text{ in } \Omega.$$

For any relatively compact subdomain $\omega \Subset \Omega$, we set

$$(1.8) \quad m(\omega) = \int_{\Omega} e^w dx, \quad \ell(\partial\omega) = \int_{\partial\omega} e^{\frac{w}{2}} ds.$$

The following inequality is the by now classical [2] Bol's isoperimetric inequality:

Theorem B. *Suppose that Ω is an open, bounded and simply-connected domain and let $w \in C^2(\Omega)$ satisfy (1.7). Assume*

$$(1.9) \quad \int_{\Omega} e^w dx \leq 8\pi.$$

Then $2\ell(\partial\omega)^2 \geq m(\omega)(8\pi - m(\omega))$ for any relatively compact subdomain $\omega \Subset \Omega$.

Here and in the rest of this paper the notation $\omega \Subset \Omega$ will be always intended to mean that ω is a relatively compact subdomain of Ω .

For the sake of completeness we remark that Theorem A and the corresponding versions of Corollaries 1.1 and 1.3 on simply connected domains has been generalized in [7] to cover the case where Dirac data are included in (1.1). In particular, a version of Theorem B suitable to be applied to that singular case, as well as the corresponding uniqueness result on simply connected domains has been obtained in [6] (see also [31]). For the corresponding existence and/or uniqueness questions on \mathbb{R}^2 or on the flat two-torus we refer the reader to [28], [33], [35].

We note that the assumption of simply-connectedness of Ω in Theorem B is necessary. See the end of section 2 below for a counterexample to the Bol's inequality in case Ω is an annulus. Actually our counterexample also shows that the inequality may fail when solutions of (1.7) share some superharmonic part in the "hole" of the annulus.

Clearly, if u solves (1.1) then $w = u - \log \int_{\Omega} e^u - \log \rho$ satisfies (1.7) with the equality sign. This is way we extend Theorem B to the case where Ω is multiply-connected and solutions of (1.7) take constant values on $\partial\Omega$.

Theorem 1.3. *Suppose that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (1.7) with $w = c$ on $\partial\Omega$, for some constant value $c \in \mathbb{R}$. Then $2\ell(\partial\omega)^2 \geq m(\omega)(8\pi - m(\omega))$ for any subdomain $\omega \Subset \Omega$. Furthermore, if ω is not simply-connected, then the inequality is strict.*

With the aid of the Bol's inequality, then Theorem 1.2 is proved by following the arguments due to Chang-Chen and the second author in [13]. However it seems that this procedure is not well-known and, in particular, there is a subtle point which also requires some modification exactly for the case $\rho = 8\pi$. We will therefore provide a complete proof of it in section 3.

A general remark is in order at this point.

Remark 1.3. *Actually the proof of Theorem 1.1 requires an estimate for the sign of a first eigenvalue and a uniqueness result suitable to be applied to a larger class of equations than (1.1), see problem (2.1) in section 2. These are the content of Theorem 3.1 in section 3, a truly more general result of independent interest which calls up among other things for a new generalization of the Bol's inequality for (2.1) on multiply connected domains, see Theorem 2.1 below. The proof of Theorem 1.3 will be derived as a straightforward consequence at the very end of section 2. A similar observation holds for Theorem 1.1 and Corollaries 1.2, 1.3 and 1.4 which has not been discussed in this introduction in their full generality to avoid technicalities. We refer to Theorem 4.1, Corollaries 4.1, 4.2 and 4.3 and Theorem 4.2 for further details concerning this point.*

As a by product of our results, we are able to solve another long-standing open problem in the rigorous statistical mechanics description of turbulent two-dimensional flows, see [10], [11] and [24]. Two main variational tools has been used so far to understand the thermodynamical equilibrium of two-dimensional turbulent Euler flows: the microcanonical and canonical variational principles, see (5.18) and (5.19) in section 5. By using the uniqueness result in [46] and other results already obtained in [10], in [11] the authors were able to establish the equivalence of these two variational principles, that is, the fact that they predict exactly the same thermodynamic. This result was achieved under certain assumptions (see Proposition 3.3 in [11]), one of which being the simply-connectedness of Ω . It seems that this restriction was entirely due to the fact that uniqueness was known only on simply-connected domains. Therefore, as a corollary of Theorem 1.2, we are able to fill this gap and obtain the equivalence of microcanonical and canonical ensembles on multiply-connected domains as well, see Theorem 5.4 in section 5 below. Actually, Theorem 1.1 and Corollary 1.4 provide an answer to another problem arising in [11], see Theorem 5.3. We refer to section 5 for further details concerning this point.

This paper is organized as follows. In section 2, the Bol's inequality for multiply-connected domains is proved together with the above mentioned counterexample. We will provide the proof of Theorem 1.2 in section 3. A more general version of Theorem 1.1 and Corollaries 1.2, 1.3 and 1.4 will be proved in section 4. Finally section 5 is devoted to the statistical mechanics applications.

2. THE BOL'S INEQUALITY ON MULTIPLY CONNECTED DOMAINS

Let Ω be an open, bounded and multiply-connected domain of class C^1 . In this section we let $\overline{\Omega}^*$ be the closure of the union of the bounded components of $\mathbb{R}^2 \setminus \partial\Omega$ and $\Omega^* = \overline{\Omega}^* \setminus \partial\overline{\Omega}^*$. Clearly, $\Omega \subseteq \Omega^*$ and $\Omega^* \equiv \Omega$ if and only if Ω is simply-connected.

Actually the proof of Theorem 1.1 requires a uniqueness result suitable to be applied to a larger class of equations than (1.1). Therefore we consider the more general problem

$$(2.1) \quad \begin{cases} \Delta u + \rho \frac{h(x)e^u}{\int_{\Omega} h(x)e^u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, here and in the rest of this paper, we assume that

$$(2.2) \quad \begin{aligned} &h(x) \text{ is strictly positive and Lipschitz continuous in } \overline{\Omega}^* \\ &\text{and } \log h(x) \text{ is subharmonic in } \overline{\Omega}^* \text{ and harmonic in } \Omega. \end{aligned}$$

Let u be a solution of (2.1). We define

$$(2.3) \quad \hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^* \setminus \Omega. \end{cases}$$

Then \widehat{u} satisfies the following inequality

$$(2.4) \quad \Delta \widehat{u} + \rho \frac{h(x)e^{\widehat{u}}}{\int_{\Omega} h(x)e^u} \geq 0 \text{ in } \Omega^*$$

in the sense of distributions. Although it is standard, we would like to provide a proof of (2.4) here for the sake of completeness.

Lemma 2.1. *Let \widehat{u} be defined by (2.3). Then \widehat{u} satisfies (2.4) in the sense of distributions.*

Proof. Let $\varphi \in C_0^2(\overline{\Omega^*})$ and $\varphi \geq 0$ in Ω^* . Then

$$\begin{aligned} \int_{\Omega^*} (\Delta \varphi) \widehat{u} dx &= \int_{\Omega} (\Delta \varphi) u dx \\ &= \int_{\Omega} \varphi (\Delta u) dx - \int_{\partial \Omega} \varphi(x) \frac{\partial u}{\partial \nu}(x) d\sigma, \end{aligned}$$

where $\nu(x)$ is the outer unit normal on $\partial \Omega$ at $x \in \partial \Omega$. Since $u(x) > 0$ in Ω , then $\frac{\partial u}{\partial \nu}(x) < 0$ on $\partial \Omega$ by the strong maximum principle. Hence

$$\begin{aligned} &\int_{\Omega^*} (\Delta \varphi) \widehat{u} dx + \frac{\rho}{\int_{\Omega} h e^u} h e^{\widehat{u}} \varphi dx \\ &\geq \int_{\Omega} \varphi \left(\Delta u + \rho \frac{h e^u}{\int_{\Omega} h e^u} \right) dx - \int_{\partial \Omega} \varphi(x) \frac{\partial u}{\partial \nu}(x) d\sigma \geq 0. \end{aligned}$$

□

Putting $\beta = \log \rho - \log \int_{\Omega} h(x) e^{u(x)}$ and

$$\begin{aligned} v(x) &= u(x) + \log h(x) + \log \rho - \log \int_{\Omega} h(x) e^{u(x)} \\ (2.5) \quad &= u(x) + \log h(x) + \beta, \quad x \in \Omega, \end{aligned}$$

we see that, as a consequence of (2.1) and (2.2), $v(x)$ satisfies

$$(2.6) \quad \Delta v + e^v = 0 \text{ in } \Omega,$$

in the sense of distributions. Let ω be a subdomain of Ω and set

$$m(\omega) = \int_{\omega} e^v dx, \quad \ell(\partial \omega) = \int_{\partial \omega} e^{\frac{v}{2}} ds.$$

Theorem 2.1. *Suppose that Ω is an open, bounded and multiply-connected domain of class C^1 and u is a solution of (2.1) with $\rho \leq 8\pi$. Then $2\ell(\partial \omega)^2 \geq m(\omega)(8\pi - m(\omega))$ for any subdomain $\omega \Subset \Omega$. Furthermore, if ω is not simply-connected, then the inequality is strict.*

We extend $v(x)$ on Ω^* by defining

$$\widehat{v}(x) = \widehat{u}(x) + \log h(x) + \beta, \quad x \in \Omega^*.$$

As a consequence of Lemma 2.1 and (2.2), \widehat{v} satisfies

$$(2.7) \quad \Delta \widehat{v} + e^{\widehat{v}} \geq 0 \text{ in } \Omega^*$$

in the sense of distributions. Clearly Theorem B could be applied on any simply-connected subdomain $\omega_0 \Subset \Omega^*$ such that $\int_{\omega_0} e^{\widehat{v}} dx \leq 8\pi$, whenever

$\widehat{v} \in C^2(\Omega^*)$. In our case however \widehat{v} is just Lipschitz continuous in Ω^* , which is why we need the following:

Lemma 2.2. *Put*

$$\widehat{m}(\omega) = \int_{\omega} e^{\widehat{v}} dx, \quad \widehat{\ell}(\partial\omega) = \int_{\partial\omega} e^{\frac{\widehat{v}}{2}} ds,$$

and let $\omega \subseteq \omega_0 \Subset \Omega^*$, where ω_0 is a simply-connected domain. If $\widehat{m}(\omega_0) \leq 8\pi$, then

$$(2.8) \quad 2\widehat{\ell}(\partial\omega)^2 \geq \widehat{m}(\omega)(8\pi - \widehat{m}(\omega)).$$

Proof. Without loss of generality, we may assume $\widehat{m}(\omega_0) < 8\pi$. Let $\psi_{\varepsilon}(x)$ be a suitable mollifier, that is, $C_0^{\infty}(\mathbb{R}^2) \ni \psi_{\varepsilon}(x) \geq 0$, $\psi_{\varepsilon}(x) = 0$ for $|x| \geq \varepsilon$ and $\int_{\mathbb{R}^2} \psi_{\varepsilon}(x) = 1$. Set $v_{\varepsilon}(x) = (\psi_{\varepsilon} * \widehat{v})(x)$ to be the standard convolution.

Thus, by using (2.7) we obtain

$$\Delta v_{\varepsilon}(x) + (\psi_{\varepsilon} * e^{\widehat{v}})(x) \geq 0 \quad \text{in } \Omega^*,$$

in the sense of distributions. Since $v_{\varepsilon}(x)$ uniformly converges to $\widehat{v}(x)$ as $\varepsilon \rightarrow 0^+$, then there exists a small constant $\delta(\varepsilon) > 0$ such that

$$(1 + \delta(\varepsilon))e^{v_{\varepsilon}(x)} \geq \psi_{\varepsilon}(x) * e^{\widehat{v}(x)} \quad \text{for } x \in \overline{\omega_0}.$$

Therefore $v_{\varepsilon}(x)$ also satisfies

$$\Delta v_{\varepsilon}(x) + (1 + \delta(\varepsilon))e^{v_{\varepsilon}(x)} \geq 0 \quad \text{in } \overline{\omega_0}.$$

For any ε small enough we have

$$\int_{\overline{\omega_0}} (1 + \delta(\varepsilon))e^{v_{\varepsilon}(x)} dx < 8\pi.$$

Let $m_{\varepsilon}(\omega)$ and $\ell_{\varepsilon}(\partial\omega)$ be defined in the usual way in terms of the metric $(1 + \delta(\varepsilon))e^{v_{\varepsilon}(x)}$ and $\sqrt{1 + \delta(\varepsilon)}e^{\frac{v_{\varepsilon}(x)}{2}}$ respectively. Then Theorem B implies

$$2\ell_{\varepsilon}^2(\partial\omega) \geq m_{\varepsilon}(\omega)(8\pi - m_{\varepsilon}(\omega)),$$

and we obtain (2.8) by passing to the limit as $\varepsilon \rightarrow 0^+$. \square

The Proof of Theorem 2.1.

Once the Bol's inequality has been established, then the fact that it is strict for domains which are not simply-connected can be proved by arguing exactly as in [13]. Since there is nothing new concerning this point we refer the reader to that paper for further details.

Next, observe that if ω is simply-connected then the conclusion easily follows from Theorem B. Therefore we assume without loss of generality that ω is multiply-connected and first consider the case where $\partial\omega$ does not bound neither one simply-connected subdomain of Ω . In this situation, each bounded component of $\mathbb{R}^2 \setminus \partial\omega$ contains at least one bounded component of $\mathbb{R}^2 \setminus \partial\Omega$. Let Ω_0 be the union of those bounded components of $\mathbb{R}^2 \setminus \partial\Omega$ which are bounded by $\partial\omega$. Let ω_0 be the union of all bounded simply-connected components of $\mathbb{R}^2 \setminus \partial\omega$. Thus $\Omega_0 \subset \omega_0$ and we define

$$\omega^* = \omega_0 \setminus \Omega_0 \cup \omega.$$

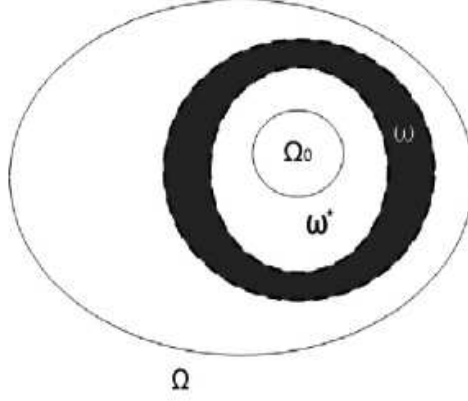


FIGURE 1

Clearly $\omega^* \subset \Omega$. See Figure 1.

In particular we see that both $\omega^* \cup \Omega_0$ and $\omega^* \cup \omega \cup \Omega_0$ are simply-connected domains. Let $\partial_0 \omega$ be the boundary of $\omega^* \cup \Omega_0$ and $\partial_1 \omega = \partial \omega \setminus \partial \omega^*$. Then $\partial_1 \omega = \partial(\omega^* \cup \omega \cup \Omega_0)$ and we have

$$\partial \omega = \partial_1 \omega \cup \partial_0 \omega.$$

Next we discuss three cases separately.

Case 1. $\widehat{m}(\omega^* \cup \Omega_0) \geq 8\pi$.

Since $u(x) > 0$ in Ω , we have

$$\begin{aligned} 2\ell(\partial_0 \omega)^2 &= 2 \left(\int_{\partial_0 \omega} e^{\frac{u(x)}{2} + \frac{1}{2} \log h(x)} e^{\frac{\beta}{2}} ds \right)^2 \\ &\geq 2 \left(e^{\frac{\beta}{2}} \int_{\partial_0 \omega} e^{\frac{1}{2} \log h(x)} ds \right)^2. \end{aligned}$$

Since $\log h(x)$ is subharmonic in Ω^* and $\partial_0 \omega = \partial(\omega^* \cup \Omega_0)$, then

$$\log h(x) \leq g(x), \quad x \in \omega^* \cup \Omega_0,$$

where $g(x)$ is the harmonic (in $\omega^* \cup \Omega_0$) function which also satisfies

$$g(x) = \log h(x), \quad x \in \partial_0 \omega.$$

Since $\omega^* \cup \Omega_0$ is simply-connected, by the Nehari's inequality [40], we have

$$\left(\int_{\partial_0 \omega} e^{\frac{1}{2} \log h(x)} ds \right)^2 = \left(\int_{\partial_0 \omega} e^{\frac{g(x)}{2}} ds \right)^2 \geq 4\pi \int_{\omega^* \cup \Omega_0} e^{g(x)} dx.$$

Hence,

$$\begin{aligned}
 (2.9) \quad 2\ell(\partial_0\omega)^2 &\geq 8\pi \int_{\omega^* \cup \Omega_0} e^{g(x)+\beta} dx \geq 8\pi \int_{\omega^* \cup \Omega_0} e^{\log h(x)+\beta} dx \\
 &> 8\pi \int_{\Omega_0} h(x)e^\beta dx = 8\pi \widehat{m}(\Omega_0)
 \end{aligned}$$

Since $\widehat{m}(\omega^* \cup \Omega_0) \geq 8\pi$, we have

$$\widehat{m}(\Omega_0) \geq 8\pi - \widehat{m}(\omega^*) \equiv 8\pi - m(\omega^*),$$

and then (2.9) yields

$$(2.10) \quad 2\ell^2(\partial_0\omega) \geq 8\pi(8\pi - m(\omega^*)).$$

Since $m(\omega \cup \omega^* \cup \Omega_0) > m(\omega^* \cup \Omega_0) \geq 8\pi$, the same argument with minor modifications can be used to obtain

$$(2.11) \quad 2\ell^2(\partial_1\omega) \geq 8\pi(8\pi - m(\omega \cup \omega^*)).$$

Hence, by using (2.10) and (2.11), we conclude that

$$\begin{aligned}
 2\ell^2(\partial\omega) &= 2(l(\partial_1\omega) + l(\partial_0\omega))^2 > 2[l^2(\partial_1\omega) + l^2(\partial_0\omega)] \\
 &\geq m(\omega^*)(8\pi - m(\omega^*)) + m(\omega \cup \omega^*)(8\pi - m(\omega \cup \omega^*)) \\
 &= (m(\omega) + m(\omega^*))(8\pi - m(\omega) - m(\omega^*)) + m(\omega^*)(8\pi - m(\omega^*)) \\
 &= m(\omega)(8\pi - m(\omega)) + m(\omega^*)(16\pi - 2m(\omega) - 2m(\omega^*)).
 \end{aligned}$$

Since

$$m(\omega) + m(\omega^*) \leq m(\Omega) \leq 8\pi,$$

then

$$2\ell^2(\partial\omega) > m(\omega)(8\pi - m(\omega)),$$

which proves Theorem 2.1 in Case 1.

Case 2. $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$ and $\widehat{m}(\omega \cup \omega^* \cup \Omega_0) \geq 8\pi$.

Since $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$ and $\omega^* \cup \Omega_0$ is simply-connected, we can apply Lemma 2.2 to obtain

$$\begin{aligned}
 (2.12) \quad 2\ell^2(\partial_0\omega) &\geq \widehat{m}(\omega^* \cup \Omega_0)(8\pi - \widehat{m}(\omega^* \cup \Omega_0)) \\
 &= (m(\omega^*) + \widehat{m}(\Omega_0))(8\pi - m(\omega^*) - \widehat{m}(\Omega_0)).
 \end{aligned}$$

As a consequence of (2.9) (which is easily seen to be satisfied in Case 2 as well) we conclude that $\ell(\partial_0\omega)$ and $\ell(\partial_1\omega)$ satisfy

$$(2.13) \quad \ell(\partial_0\omega) \geq \sqrt{4\pi\widehat{m}(\Omega_0)} \quad \text{and} \quad \ell(\partial_1\omega) \geq \sqrt{4\pi\widehat{m}(\Omega_0)}.$$

Clearly (2.11) holds in Case 2 as well and therefore it can be used together with (2.12) and (2.13) to conclude that

$$\begin{aligned}
2\ell^2(\partial\omega) &= 2[\ell^2(\partial_1\omega) + 2\ell(\partial_1\omega)\ell(\partial_0\omega) + \ell^2(\partial_0\omega)] \\
&\geq 8\pi(8\pi - m(\omega) - m(\omega^*)) \\
&\quad + (m(\omega^*) + \widehat{m}(\Omega_0))(8\pi - m(\omega^*) - \widehat{m}(\Omega_0)) + 16\pi\widehat{m}(\Omega_0) \\
&= 8\pi(8\pi - m(\omega)) - m^2(\omega^*) + \widehat{m}(\Omega_0)(24\pi - 2m(\omega^*) - \widehat{m}(\Omega_0)) \\
&= m(\omega)(8\pi - m(\omega)) + [(8\pi - m(\omega))^2 - m^2(\omega^*)] \\
&\quad + \widehat{m}(\Omega_0)(24\pi - 2m(\omega^*) - \widehat{m}(\Omega_0)).
\end{aligned}$$

Since $m(\omega) + m(\omega^*) \leq m(\Omega) \leq 8\pi$, then

$$m(\omega^*) \leq 8\pi - m(\omega).$$

Moreover since $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$, then

$$2m(\omega^*) + 2\widehat{m}(\Omega_0) < 16\pi.$$

Hence $2\ell^2(\partial\omega) > m(\omega)(8\pi - m(\omega))$ which proves Theorem 2.1 in Case 2 as well.

Case 3. $\widehat{m}(\omega \cup \omega^* \cup \Omega_0) < 8\pi$.

Since $\omega \cup \omega^* \cup \Omega_0$ is simply-connected, by applying Lemma 2.2 to ω , we have

$$2\ell(\partial\omega)^2 \geq m(\omega)(8\pi - m(\omega)),$$

which concludes the proof of Theorem 2.1 in case the interior of $\partial\omega$ does not contain neither one simply-connected subdomain of Ω .

Now suppose that $\partial\omega$ bounds some simply-connected subdomains of Ω which we denote by $\omega_1, \dots, \omega_k$ with $k \geq 1$. Then $\omega \cup \overline{\omega_1} \cup \dots \cup \overline{\omega_k}$ is connected in Ω and its boundary does not bound any simply-connected component in Ω . In particular

$$\partial\omega = \partial(\omega \cup \omega_1 \cup \dots \cup \omega_k) \setminus \bigcup_{j=1}^k (\partial\omega_j).$$

By applying the previous result to $\omega \cup \omega_1 \cup \dots \cup \omega_k$ and Lemma 2.2 (or either Theorem B) to the domains $\omega_j, 1 \leq j \leq k$, we obtain

$$\begin{aligned}
(2.14) \quad 2\ell^2(\partial(\omega \cup \omega_1 \cup \dots \cup \omega_k)) &\geq m(\omega \cup \omega_1 \cup \dots \cup \omega_k) \\
&\quad [8\pi - m(\omega \cup \omega_1, \dots, \cup \omega_k)],
\end{aligned}$$

$$(2.15) \quad 2\ell^2(\partial\omega_j) \geq m(\omega_j)(8\pi - m(\omega_j)).$$

If $k = 1$, by using (2.14) and (2.15) we have

$$\begin{aligned}
2\ell(\partial\omega)^2 &> 2\ell(\partial(\omega \cup \omega_1))^2 + 2\ell(\partial\omega_1)^2 \\
&\geq m(\omega \cup \omega_1)(8\pi - m(\omega \cup \omega_1)) + m(\omega_1)(8\pi - m(\omega_1)) \\
&= m(\omega)(8\pi - m(\omega_1)) + m(\omega_1)(16\pi - 2m(\omega_1) - 2m(\omega)) \\
&\geq m(\omega)(8\pi - m(\omega)).
\end{aligned}$$

We omit the details of the case $k > 1$ which is worked out by similar arguments. \square

At this point we are ready to provide the following:

Proof of Theorem 1.3

Since $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (1.7) and $w = c$ on $\partial\Omega$ then, by arguing exactly as in Lemma 2.1, it is easy to check that setting $u = w - c$ and \hat{u} to be the corresponding (vanishing) extension to Ω^* as defined in (2.3), then it satisfies

$$\Delta \hat{u} + e^c e^{\hat{u}} \geq 0 \text{ in } \Omega^*,$$

in the sense of distributions. Hence $\hat{v} = \hat{u} + c$ is Lipschitz continuous in Ω^* and satisfies (2.7) in the sense of distributions. The rest of the proof of Theorem 2.1 (including of course that of Lemma 2.2) works without any further modification and the desired conclusion follows. \square

We conclude this section with an example which shows at the same time that if Ω is not simply-connected then Theorem B fails and if $\log h$ cannot be extended to a subharmonic function in Ω^* (see (2.2)) then Theorem 2.1 fails.

Example [*Failure of the Bol's inequality on multiply-connected domains*]

For any $-1 < \alpha < 0$ and $a > 0$ let us define

$$v_\alpha(x) = \log \left(\frac{8(1+\alpha)^2 a^2 |x|^{2\alpha}}{(1+a^2|x|^{2(1+\alpha)})^2} \right), \quad x \in \mathbb{R}^2.$$

Observe that v_α satisfies $-\Delta v_\alpha = e^{v_\alpha} - 4\pi\alpha\delta_{p=0}$ in the sense of distributions in \mathbb{R}^2 and in particular $-\Delta v_\alpha = e^{v_\alpha}$ classically in $\mathbb{R}^2 \setminus \{0\}$. Since $v_\alpha \simeq 2\alpha \log |x|$ as $x \rightarrow 0$ it is clear that its leading term is superharmonic near the origin. For each $0 < s < t < +\infty$ let us set

$$A_{s,t} = \{x \in \mathbb{R}^2 \mid s < |x| < t\},$$

and for $0 < R_1 < r_1 < r_2 < R_2 < +\infty$ let us define

$$\Omega := A_{R_1, R_2}, \quad \omega = A_{r_1, r_2}.$$

We have

$$\begin{aligned} \ell(\omega) &= \int_{\{|x|=r_1\} \cup \{|x|=r_2\}} \frac{\sqrt{8}(1+\alpha)a|x|^\alpha}{1+a^2|x|^{2(1+\alpha)}} ds = \\ &= \frac{2\pi\sqrt{8}(1+\alpha)ar_1^{\alpha+1}}{1+a^2r_1^{2(1+\alpha)}} + \frac{2\pi\sqrt{8}(1+\alpha)ar_2^{\alpha+1}}{1+a^2r_2^{2(1+\alpha)}}, \end{aligned}$$

and

$$\begin{aligned} m(\omega) &= \int_{A_{r_1, r_2}} \frac{8(1+\alpha)^2 a^2 |x|^{2\alpha}}{(1+a^2|x|^{2(1+\alpha)})^2} dx = \\ &= 8\pi(1+\alpha) \left(\frac{1}{1+a^2r_1^{2(1+\alpha)}} - \frac{1}{1+a^2r_2^{2(1+\alpha)}} \right). \end{aligned}$$

Clearly the last identity implies

$$\int_{\Omega} e^{v\alpha} < \int_{\mathbb{R}^2} e^{v\alpha} = 8\pi(1+\alpha) < 8\pi.$$

Therefore, letting $0 < R_1 < r_1 \searrow 0$, we obtain

$$2\ell^2(\omega) = 64\pi^2(1+\alpha)^2 \frac{a^2 r_2^{2(1+\alpha)}}{\left(1 + a^2 r_2^{2(1+\alpha)}\right)^2} + o(1),$$

and

$$\begin{aligned} m(\omega)(8\pi(1+\alpha) - m(\omega)) = \\ 64\pi^2(1+\alpha)^2 \left(\frac{a^2 r_2^{2(1+\alpha)}}{\left(1 + a^2 r_2^{2(1+\alpha)}\right)} + o(1) \right) \left(\frac{1}{\left(1 + a^2 r_2^{2(1+\alpha)}\right)} + o(1) \right). \end{aligned}$$

We readily conclude that as $0 < R_1 < r_1 \searrow 0$ it holds

$$2\ell^2(\omega) = m(\omega)(8\pi(1+\alpha) - m(\omega)) + o(1),$$

and then, for any fixed $-1 < \alpha < 0$ and any $0 < R_1 < r_1$ small enough we see that the inequality

$$2\ell^2(\omega) = m(\omega)(8\pi - m(\omega)) + 8\pi\alpha m(\omega) + o(1) \leq$$

$$m(\omega)(8\pi - m(\omega)) + 4\pi\alpha m(\omega) < m(\omega)(8\pi - m(\omega)),$$

holds. As a consequence the Bol's inequality does not hold in the situation at hand.

3. SYMMETRIZATION AND UNIQUENESS

The main theorem in this section is the following.

Theorem 3.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain of class C^1 and $h(x)$ satisfies (2.2). Then, for any $\rho \leq 8\pi$, there exists at most one solution for problem (2.1). In particular the first eigenvalue of the linearized problem for (2.1) is strictly positive for any $\rho \leq 8\pi$.*

As a matter of fact, the proof of Theorem 3.1 can be worked out by using a rearrangement argument together with the improved (see Theorem 2.1) Bol's inequality. As mentioned above, the first proof for $\rho < 8\pi$ on simply-connected domains was given by Suzuki in [46]. That argument was improved in [13] to cover the case $\rho \leq 8\pi$ and later in [6] where the more general situation of singular (Dirac) data was considered as well. However that rearrangement argument seems not to be well-known and in particular the argument used to handle the more subtle case, that is $\rho = 8\pi$, has to be modified with respect to the one adopted in [13], [6]. Hence we will present a complete proof here for the sake of completeness.

Proof. Since Theorem 3.1 is well known for Ω simply-connected, with the unique exception of the incoming statement of Lemma 3.1 we will assume that Ω is not simply-connected for the rest of this section. In any case Ω will be always assumed to be open, bounded and of class C^1 .

For any such Ω , and for fixed $V \in L^\infty(\Omega)$, we will say that $\lambda_k = \lambda_k(V, \Omega)$ is the k -th eigenvalue of $\Delta + V$ if there exists $\psi_k \in H_0^1(\Omega)$ such that

$$-\Delta\psi_k - V\psi_k = \lambda_k V\psi_k \quad \text{in } \Omega.$$

We begin with the following Lemma of independent interest. The first part is well known see [2], [46] and more recently [6], while the second is the novel generalization of those results to the case where Ω is multiply-connected and of class C^1 .

Lemma 3.1.

(I) *Let w satisfy (1.7) and Ω be simply-connected. Then $\lambda_1(e^w, \Omega) > 0$ whenever $\int_\Omega e^w \leq 4\pi$ while $\lambda_2(e^w, \Omega) > 0$ whenever $\int_\Omega e^w \leq 8\pi$.*

(II) *Let v take the form (2.5) and therefore satisfy (2.6) on the multiply-connected domain Ω .*

Then $\lambda_1(e^v, \Omega) > 0$ whenever $\rho \leq 4\pi$ while $\lambda_2(e^v, \Omega) > 0$ whenever $\rho \leq 8\pi$.

Proof. As mentioned above (I) is well known, see for example [6] for a detailed proof.

We will first prove the assertion (II) concerning $\lambda_2 \equiv \lambda_2(e^v, \Omega)$. We argue by contradiction and suppose that $\lambda_2 \leq 0$. Then there exists $K \leq 1$ and a second eigenfunction φ satisfying

$$(3.1) \quad \begin{cases} \Delta\varphi + Ke^v\varphi = 0 & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

Let $\Omega^+ = \{x \in \Omega | \varphi(x) > 0\}$. We want to prove that

$$(3.2) \quad \int_{\Omega^+} e^{v(x)} dx \geq 4\pi.$$

Set $U(x) = -2\log(1 + \frac{1}{8}|x|^2)$, which is an entire solution of

$$(3.3) \quad \Delta U + e^U = 0 \quad \text{in } \mathbb{R}^2.$$

For any $t > 0$, set $\Omega_t = \{x \in \Omega | \varphi(x) > t\} \Subset \Omega$, and let $r(t)$ be uniquely defined by the equality

$$(3.4) \quad \int_{B_{r(t)}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx,$$

where $B_{r(t)}$ is the open ball of center O and radius $r(t)$. Since φ is smooth, $r(t)$ is a strictly decreasing and continuous function of $t \in [0, \max_\Omega \varphi]$. Let

$$(3.5) \quad \varphi^*(r) = \sup_{t>0} \{t | v < r(t)\}.$$

Thus, for $t \in [0, \max_{\Omega} \varphi]$, the equalities

$$(3.6) \quad \varphi^*(r(t)) = t \quad \text{and} \quad r(\varphi^*(r)) = r$$

hold. Clearly (3.4) implies

$$(3.7) \quad \int_{\{\varphi^* > t\}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx,$$

and

$$(3.8) \quad \int_{B_{R_0}} e^{U(x)} (\varphi^*)^2 dx = \int_{\Omega_+} e^{v(x)} \varphi^2 dx,$$

where $R_0 = r(0)$. To derive a contradiction, we use the coarea formulas

$$(3.9) \quad \begin{aligned} -\frac{d}{dt} \int_{\Omega_t} |\nabla \varphi|^2 dx &= \int_{\partial \Omega_t} |\nabla \varphi| ds, \quad \text{and} \\ -\frac{d}{dt} \int_{\Omega_t} e^{v(x)} dx &= \int_{\partial \Omega_t} \frac{e^v}{|\nabla \varphi|} ds \end{aligned}$$

which hold simultaneously for almost any t . Since $\int_{\Omega} e^v dx = \rho \leq 8\pi$, then (3.9), the Cauchy-Schwarz inequality and Theorem 2.1 together imply

$$\begin{aligned} (3.10) \quad & -\frac{d}{dt} \int_{\Omega_t} |\nabla \varphi|^2 dx = \int_{\{\varphi=t\}} |\nabla \varphi| ds \\ & \geq \left(\int_{\{\varphi=t\}} e^{v/2} ds \right)^2 \left(\int_{\{\varphi=t\}} \frac{e^v}{|\nabla \varphi|} ds \right)^{-1} \\ & = \ell^2(\{\varphi=t\}) \left(-\frac{d}{dt} \int_{\Omega_t} e^{v(x)} dx \right)^{-1} \\ & \geq \frac{1}{2} \left(8\pi - \int_{\Omega_t} e^{v(x)} dx \right) \left(\int_{\Omega_t} e^{v(x)} dx \right) \left(-\frac{d}{dt} \int_{\Omega_t} e^{v(x)} dx \right)^{-1} \\ & = \frac{1}{2} \left(8\pi - \int_{\{\varphi^* > t\}} e^{U(x)} dx \right) \left(\int_{\{\varphi^* > t\}} e^{U(x)} dx \right) \left(-\frac{d}{dt} \int_{\{\varphi^* > t\}} e^{U(x)} dx \right)^{-1}. \end{aligned}$$

The same computation for $\nabla \varphi^*$ yields,

$$\begin{aligned} (3.11) \quad & -\frac{d}{dt} \int_{\{\varphi^* > t\}} |\nabla \varphi^*|^2 dx \\ & = \frac{1}{2} \left(8\pi - \int_{\{\varphi^* > t\}} e^{U(x)} dx \right) \left(\int_{\{\varphi^* > t\}} e^{U(x)} dx \right) \left(-\frac{d}{dt} \int_{\{\varphi^* > t\}} e^{U(x)} dx \right)^{-1}, \end{aligned}$$

for the same values of t , except possibly for a set of null measure. Therefore,

$$(3.12) \quad -\frac{d}{dt} \int_{\{\varphi > t\}} |\nabla \varphi|^2 dx \geq -\frac{d}{dt} \int_{\{\varphi^* > t\}} |\nabla \varphi^*|^2 dx$$

holds for almost any t . By integrating the above inequality, we obtain

$$(3.13) \quad \int_{B_{R_0}} |\nabla \varphi^*|^2 dx \leq \int_{\Omega_+} |\nabla \varphi|^2 dx.$$

On the other side, (3.1) implies

$$\int_{\Omega_+} |\nabla \varphi|^2 dx = K \int_{\Omega_+} e^{v(x)} \varphi^2 dx,$$

so that (3.13) yields

$$\int_{B_{R_0}} |\nabla \varphi^*|^2 dx - \int_{B_{R_0}} e^{U(x)} (\varphi^*)^2 dx \leq \int_{\Omega_+} |\nabla \varphi|^2 dx - \int_{\Omega_+} e^{v(x)} \varphi^2 dx \leq 0.$$

Thus, the first eigenvalue of $\Delta + e^{U(x)}$ is nonpositive. By a straightforward computation, the function $z(r) = \frac{8-r^2}{8+r^2}$ satisfies

$$(3.14) \quad \Delta z + e^{U(r)} z = 0 \quad \text{in } \mathbb{R}^2.$$

Since $z(r) \geq 0$ for $r \leq \sqrt{8}$, then the first eigenvalue of $\Delta + e^{U(x)}$ for $B_{\sqrt{8}}$ is equal to zero. Hence

$$(3.15) \quad \sqrt{8} \leq R_0,$$

and (3.2) readily follows since we have (see (3.7))

$$\int_{\Omega_+} e^{v(x)} dx = \int_{B_{R_0}} e^{U(x)} dx \geq \int_{B_{\sqrt{8}}} e^{U(x)} dx = 4\pi.$$

Next, let $\Omega_- = \{x | \varphi(x) < 0\}$. By using the same argument we obtain

$$\int_{\Omega_-} e^{v(x)} dx \geq 4\pi,$$

and

$$\rho = \int_{\Omega} e^{v(x)} dx = \int_{\Omega_+} e^{v(x)} dx + \int_{\Omega_-} e^{v(x)} dx \geq 8\pi.$$

Of course, this is already a contradiction whenever $\rho < 8\pi$.

In case $\rho = 8\pi$ then (3.2) turns out to be an equality i.e.

$$\int_{\Omega_+} e^{v(x)} dx = 4\pi = \int_{\Omega_-} e^{v(x)} dx,$$

and all the inequalities in (3.10) are equalities. In particular, for $t > 0$ and $s < 0$ the domains $\{x \in \Omega | \varphi(x) > t\}$ and $\{x \in \Omega | \varphi(x) < s\}$ are simply-connected (by Theorem 2.1) and in particular $|\nabla \varphi(x)| e^{-v(x)} = |\nabla \varphi(y)| e^{-v(y)}$ whenever $\varphi(x) = \varphi(y) \neq 0$, because of the equality in the Cauchy-Schwarz inequality.

Therefore, passing to the limit, the equality above holds also for the set $\{x | \varphi(x) = 0\}$, i.e.,

$$(3.16) \quad |\nabla \varphi(x)| e^{-v(x)} = |\nabla \varphi(y)| e^{-v(y)} \quad \text{whenever } \varphi(x) = \varphi(y) = 0.$$

Since both Ω^+ and Ω^- are simply-connected while Ω is not, then the nodal line $\{x \in \Omega | \varphi(x) = 0\}$ must intersect $\partial\Omega$ at some point (say) $p_0 \in \partial\Omega$. Clearly, $\nabla \varphi(p_0) = 0$, and (3.16) yields

$$\nabla \varphi(x) = 0 \quad \text{for } x \in \{x \in \overline{\Omega} | \varphi(x) = 0\},$$

which in view of the strong maximum principle implies $\varphi \equiv 0$. This is the desired contradiction for $\rho = 8\pi$ which concludes the proof of that part of (II) which is concerned with $\lambda_2 \equiv \lambda_2(e^v, \Omega)$.

At this point however the assertion concerning λ_1 for $\rho < 4\pi$ is easily worked out by arguing as we did above via rearrangement and just replacing Ω^+ with Ω . Finally, in case $\rho = 4\pi$, we conclude once more that all the inequalities in (3.10) are equalities, which is impossible in view of Theorem 2.1 and the fact that Ω is multiply-connected. \square

It turns out that the same argument used in the proof of (I) in [6] as well as the one used in the proof of (II) above show that the following useful result holds:

Lemma 3.2.

Let either w satisfy (1.7) and Ω be simply connected or v take the form (2.5) and satisfy (2.6) on the multiply-connected domain Ω and set either $V = e^w$ or $V = e^v$ respectively.

Assume moreover that $\int_{\Omega} V \leq 8\pi$ and that on some subdomain $\omega \Subset \Omega$ there exists $\psi \in C_0^2(\omega) \cap C^0(\overline{\omega})$ which satisfies

$$-\Delta\psi - V\psi \leq 0 \quad \text{in } \omega.$$

If $\psi > 0$ in ω , then $\int_{\omega} V \geq 4\pi$.

Next, let us prove Theorem 3.1.

The Proof of Theorem 3.1.

The main point in the proof of Theorem 3.1 is to show that the linearized operator for (2.1) is non-singular whenever $\rho \leq 8\pi$. Once this fact is known, the proof can be completed by known arguments, see for example [6]. We refer the reader to that paper for further details concerning this point.

We argue by contradiction and suppose that φ is a solution of the linearized problem for (2.1) with $\rho \leq 8\pi$. Then φ satisfies

$$\Delta\varphi + \frac{\rho h e^u \varphi}{\int_{\Omega} h e^u} - \frac{\rho h e^u (\int_{\Omega} h e^u \varphi)}{(\int_{\Omega} h e^u)^2} = 0 \quad \text{in } \Omega,$$

$$\varphi|_{\partial\Omega} = 0.$$

By adding $\frac{-\int_{\Omega} h e^u \varphi}{\int_{\Omega} h e^u}$ to φ we come up with a new function (still denoted by φ) which satisfies

$$(3.17) \quad \begin{cases} \Delta\varphi + \frac{\rho h e^u \varphi}{\int_{\Omega} h e^u} = 0 & \text{in } \Omega, \\ \int_{\Omega} |\nabla\varphi|^2 = 1 \\ \int_{\Omega} \rho h e^u \varphi = 0 \quad \text{and } \varphi = c \text{ on } \partial\Omega, \end{cases}$$

where c is a constant. Since $\int \rho h e^u \varphi = 0$, φ changes sign in Ω . If $c = 0$, then the second eigenvalue of $\Delta + \frac{\rho h e^u}{\int \rho h e^u}$ would be non-positive, in contradiction with Lemma 3.1(II). Hence, we may assume that $c \neq 0$ and in particular, without loss of generality, that in fact $c < 0$. We define

$$\tilde{\Omega}^+ = \{x \in \Omega \mid \varphi(x) > c\},$$

and divide the proof in two cases.

Case 1: $\tilde{\Omega}^+ = \Omega$.

We argue as in the proof of Lemma 3.1 and apply the rearrangement argument to obtain a contradiction.

Set $\Omega_t = \{x \in \Omega \mid \varphi(x) > t\}$ for $t \in [c, \max_{\bar{\Omega}} \varphi]$ and $U(x)$ and φ^* as in Lemma

3.1. Thus, we have

$$(3.18) \quad \int_{\{\varphi^* > t\}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx,$$

$$(3.19) \quad \int_{B_{R_0}} e^{U(x)} \varphi^* dx = \int_{\Omega} e^{v(x)} \varphi dx = 0,$$

$$(3.20) \quad \int_{B_{R_0}} |\nabla \varphi^*|^2 < \int_{\Omega} |\nabla \varphi|^2 dx,$$

where $R_0 = r(c)$ satisfies

$$(3.21) \quad \int_{B_{R_0}} e^{U(x)} dx = \rho \leq 8\pi,$$

and $R_0 < +\infty$ if $\rho < 8\pi$, while $R_0 = +\infty$ if $\rho = 8\pi$.

It is worth to point out that the strict inequality in (3.20) is due to the fact that Ω is not simply-connected (see Theorem 2.1).

Since $c < 0$ then is well defined $\xi_0 = r(0)$ and clearly $\varphi^*(\xi_0) = 0$. It is easy to verify that we may assume (3.2) (that is $\int_{\Omega_+} e^{v(x)} dx \geq 4\pi$) to be satisfied in the situation under consideration. Then we conclude that

$$\int_{B_{\xi_0}} e^{U(x)} dx = \int_{\{\varphi^* > 0\}} e^{U(x)} dx = \int_{\{\varphi > 0\}} e^{v(x)} dx \geq 4\pi.$$

Hence,

$$(3.22) \quad \xi_0 \geq \sqrt{8}.$$

Next, we observe that

$$(3.23) \quad \int_{\Omega} |\nabla \varphi|^2 dx = \int_{\Omega} e^v \varphi^2 dx = 1.$$

Therefore, putting

$$k_0 = \inf \left\{ \int_{B_{R_0}} |\nabla \psi|^2 dx \mid \psi(x) = \psi(|x|) \text{ and } \psi(\xi_0) = 0, \right. \\ \left. \int_{B_{R_0}} e^{U(x)} \psi(x) dx = 0, \quad \int_{B_{R_0}} e^{U(x)} \psi^2 dx = 1 \right\},$$

we see that (3.23) and (3.8), (3.19), (3.20) together imply $k_0 < 1$.

It is not difficult to see that the infimum is always achieved by some function ψ^* where ψ^* is continuous in $\overline{B_{R_0}}$ and satisfies

$$(3.24) \quad \begin{cases} \Delta\psi^* + k_0 e^{U(x)}\psi^* = 0 & \text{in } 0 < r < \xi_0 \text{ and } \xi_0 < r < R_0 \\ \int_{B_{R_0}} e^{U(x)}\psi^*(x)dx = 0, \psi^*(\xi_0) = 0 & \text{and } \psi^{*'}(R_0) = 0. \end{cases}$$

If $R_0 < +\infty$, then

$$\begin{aligned} \lim_{r \uparrow \xi_0} \psi^{*'}(r)r &= -k_0 \int_0^{\xi_0} e^{U(r)}\psi^*(r)rdr, \text{ and} \\ \lim_{r \downarrow \xi_0} \psi^{*'}(r)r &= k_0 \int_{\xi_0}^{R_0} e^{U(r)}\psi^*(r)rdr, \end{aligned}$$

the last equality being a consequence of the condition $\psi^{*'}(R_0) = 0$. Since ψ^* satisfies

$$\int_{B_{R_0}} e^{U(r)}\psi^*(r)dx = 0,$$

we conclude that

$$(3.25) \quad \lim_{r \uparrow \xi_0} \psi^{*'}(r) = \lim_{r \downarrow \xi_0} \psi^{*'}(r).$$

If $R_0 = +\infty$ and since $\psi^{*'} \in L^2(\mathbb{R}^2)$, there exists $r_n \rightarrow +\infty$ such that

$$\psi^{*'}(r_n)r_n \rightarrow 0,$$

which implies

$$\lim_{r \downarrow \xi_0} \psi^{*'}(r)r = \int_{\xi_0}^{\infty} e^{U(x)}\psi^*(r)rdr.$$

Therefore we readily verify that (3.25) holds in case $R_0 = +\infty$ as well. Hence ψ^* is of class C^2 and satisfies

$$(3.26) \quad \begin{cases} \Delta\psi^* + k_0 e^{U(x)}\psi^* = 0 & \text{in } B_{R_0}, \text{ and} \\ \int_{B_{R_0}} e^{U(x)}\psi^*dx = 0 & \text{and } \psi^*(\xi_0) = 0, \psi^{*'}(R_0) = 0 \end{cases}$$

for some $0 < k_0 < 1$.

Clearly $\psi^*(r)$ changes signs once and (3.21) to be used together with Lemma 3.2 shows that in fact it changes sign just once. Therefore we can assume without loss of generality that $\psi^*(r) > 0$ if $r < \xi_0$ and $\psi^*(r) < 0$ if $r > \xi_0$. Hence, by using the integral constraint in (3.26), we see that for each $0 < r < R_0$ it holds

$$\int_0^r e^{U(r)}\psi^*(r)rdr > \int_0^{R_0} e^{U(r)}\psi^*(r)rdr = 0.$$

Thus $\psi^{*'}(r) < 0$ for $0 < r < R_0$.

Let $z(r) = \frac{8-r^2}{8+r^2}$ be the function defined above and satisfying (3.14). Note that $z(\sqrt{8}) = 0$ and in particular that $\psi(\xi_0) = 0$ and $\xi_0 \geq \sqrt{8}$ (see (3.22)). We want to prove $\xi_0 = \sqrt{8}$.

We assume (by contradiction) that $\xi_0 > \sqrt{8}$. Then by using the equation in (3.26) and (3.14) we see that for each $\xi_0 < r < R_0$ it holds

$$(3.27) \quad \begin{aligned} & \lim_{R \rightarrow R_0} \left(\frac{\psi^*(R)}{z(R)} \right)' z^2(R) - r \left(\frac{\psi^*(r)}{z(r)} \right)' z^2(r) \\ &= (1-k) \int_r^{R_0} e^{U(s)} \psi^*(s) z(s) ds. \end{aligned}$$

In the same time we see that either $R_0 < +\infty$ and then

$$(3.28) \quad \begin{aligned} R_0 \left(\frac{\psi^*(R_0)}{z(R_0)} \right)' z^2(R_0) &= R_0 (\psi^{*'}(R_0) z(R_0) - z'(R_0) \psi^*(R_0)) \\ &= -R_0 z'(R_0) \psi^*(R_0) < 0, \end{aligned}$$

or $R_0 = +\infty$ and then

$$(3.29) \quad \lim_{R \rightarrow +\infty} R \left(\frac{\psi^*(R)}{z(R)} \right)' z^2(R) = 0.$$

Hence we can use (3.27) together with (3.28) and (3.29) to conclude that

$$\left(\frac{\psi^*(r)}{z(r)} \right)' < 0, \quad \text{if } \xi_0 < r < R_0.$$

This inequality in turn yields

$$0 = \frac{\psi^*(\xi_0)}{z(\xi_0)} > \frac{\psi^*(r)}{z(r)} > 0, \quad \text{if } \sqrt{8} < \xi_0 \leq r$$

a contradiction. Therefore we conclude that $\xi_0 \leq \sqrt{8}$ and in view of (3.22) $\xi_0 = \sqrt{8}$ as desired.

At this point, by using (3.26), we check that ψ^* is a positive eigenfunction for $\Delta + e^{U(r)}$ on the ball $B_{\sqrt{8}}$ corresponding to the eigenvalue $k_0 - 1$. Hence ψ^* must be the first eigenfunction corresponding to the first eigenvalue which, of course, is zero (its eigenfunction being z). Hence we must have $k_0 = 1$ which is the desired contradiction in Case 1.

Case 2. We assume both $\tilde{\Omega}^+ = \{x \in \Omega \mid \varphi(x) > c\}$ and $\tilde{\Omega}^- = \{x \in \Omega \mid \varphi(x) < c\}$ are not empty sets.

In this case, we set $r(t)$ for $t > c$ and $R(t)$ for $t < c$ by

$$\int_{B_{r(t)}} e^{U(x)} dx = \int_{\{\varphi > t\}} e^{v(x)} dx \quad \text{if } t > c,$$

and,

$$\int_{\mathbb{R}^2 \setminus B_{R(t)}} e^{U(x)} dx = \int_{\{\varphi < t\}} e^{v(x)} dx \quad \text{if } t < c.$$

Let $r_0 = \lim_{t \downarrow c} r(t)$ and $R_0 = \lim_{t \uparrow c} R(t)$. Of course we have

$$(3.30) \quad r_0 \leq R_0 \quad \text{and} \quad r_0 = R_0 \quad \text{if and only if} \quad \rho = 8\pi.$$

Let φ^* and $\tilde{\varphi}$ be the symmetrization of φ for the parts $\{\varphi > c\}$ and $\{\varphi < c\}$ respectively. Thus, by arguing as in Lemma 3.1, we have

$$\begin{aligned} & \int_{B_{r_0}} |\nabla \varphi^*(x)|^2 dx + \int_{B_{r_0}} e^U |\varphi^*(x)|^2 dx \\ & \leq \int_{\{\varphi > c\}} |\nabla \varphi(x)|^2 + \int_{\{\varphi > c\}} e^v |\varphi(x)|^2 dx, \\ \text{and} \\ & \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \tilde{\varphi}(x)|^2 + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U |\tilde{\varphi}(x)|^2 dx \\ & \leq \int_{\{\varphi < c\}} |\nabla \varphi(x)|^2 dx + \int_{\{\varphi < c\}} e^v |\varphi(x)|^2 dx. \end{aligned}$$

Therefore, by using (3.17), we obtain

$$\begin{aligned} (3.31) \quad & \int_{B_{r_0}} |\nabla \varphi^*(x)|^2 + \int_{B_{r_0}} e^U |\varphi^*(x)|^2 + \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \tilde{\varphi}(x)|^2 \\ & + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U |\tilde{\varphi}(x)|^2 \leq 0. \end{aligned}$$

Clearly, φ^* and $\tilde{\varphi}$ together satisfies

$$\int_{B_{r_0}} e^U \varphi^*(x) + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U \tilde{\varphi}(x) dx = \int_{\Omega} e^v \varphi(x) dx = 0.$$

Let $\xi_0 = r(0) < r_0 = r(c)$. Then $\varphi^*(\xi_0) = 0$ and we may still assume without loss of generality (3.2) (that is $\int_{\Omega_+} e^{v(x)} dx \geq 4\pi$) to be satisfied so that also (3.22) (that is $\xi_0 \geq \sqrt{8}$) holds in Case 2 as well.

$$\text{Set } H = \left\{ (\psi^*, \tilde{\psi}) \mid \psi^*(x) = \psi^*(|x|) \text{ and } \tilde{\psi}(x) = \tilde{\psi}(|x|) \right.$$

are defined in B_{r_0} and $\mathbb{R}^2 \setminus B_{R_0}$ respectively, and satisfy

$$\psi^*(\xi_0) = 0, \psi^*(r_0) = \tilde{\psi}(R_0) = c \text{ and}$$

$$\left. \int_{B_{r_0}} e^U \psi^*(x) dx + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U \tilde{\psi}(x) dx = 0 \right\},$$

and

$$k = \inf \left\{ \int_{B_{r_0}} |\nabla \psi^*|^2 + \int_{\mathbb{R}^2 \setminus B_{R_0}} |\nabla \tilde{\psi}|^2 dx \right\},$$

where the infimum is taken over the set of all $(\psi^*, \tilde{\psi}) \in H$ such that

$$\int_{B_{r_0}} e^U |\psi^*|^2 dx + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U |\tilde{\psi}(x)|^2 dx = 1.$$

Hence, by using (3.31) we obtain

$$0 < k \leq 1.$$

It is easy to see that the infimum is achieved and we denote by $(\psi^*, \tilde{\psi})$ the corresponding minimizers. By using the same arguments adopted above, we can verify that (3.25) holds in Case 2 as well. In particular we have

$$(3.32) \quad \begin{cases} \Delta \psi^* + k e^U \psi^* = 0 & \text{in } B_{r_0}, \\ \Delta \tilde{\psi} + k e^U \tilde{\psi} = 0 & \text{in } \mathbb{R}^2 \setminus B_{R_0}, \\ \psi^*(\xi_0) = 0, \psi^*(r_0) = \tilde{\psi}(R_0) = c & \text{and} \\ \int_{B_{r_0}} e^U \psi^* dx + \int_{\mathbb{R}^2 \setminus B_{R_0}} e^U \tilde{\psi}(x) dx = 0 \end{cases}$$

Since $\tilde{\psi}'(r)r \rightarrow 0$ as $r \rightarrow +\infty$, we also have

$$(3.33) \quad \psi^{*'}(r_0)r_0 = \psi^{*'}(R_0)R_0.$$

At this point we want to show $\xi_0 = \sqrt{8}$. Set $z(r) = \frac{r^2-8}{r^2+8}$.

Hence, let us assume (by contradiction) that $\xi_0 > \sqrt{8}$. Clearly we can use (3.27) to obtain

$$\begin{aligned} & r_0 \left(\frac{\psi^*(r_0)}{z(r_0)} \right)' z^2(r_0) - \xi_0 \left(\frac{\psi^*(\xi_0)}{z(\xi_0)} \right)' z^2(\xi_0) \\ &= (1-k) \int_{\xi_0}^{r_0} e^{U(s)} \psi^*(s) z(s) ds \geq 0, \end{aligned}$$

where $\psi^*(s) \leq 0$ and $z(s) \leq 0$ for $s \geq \xi_0 \geq \sqrt{8}$.

Thus, $r_0(\psi^{*'}(r_0)z(r_0) - z'(r_0)\psi^*(r_0)) \geq \xi_0\psi^{*'}(\xi_0)z(\xi_0) > 0$, where $\psi^{*'}(\xi_0) < 0$ and $z(\xi_0) < 0$. Therefore, since $z(r_0)\psi^*(r_0) > 0$, we readily obtain

$$\frac{\psi^{*'}(r_0)}{\psi^*(r_0)} > \frac{z'(r_0)}{z(r_0)}.$$

This relation can be used together with (3.33) to conclude that

$$(3.34) \quad \frac{\psi^{*'}(R_0)R_0}{\psi^*(R_0)} = \frac{\psi^{*'}(r_0)r_0}{\psi^*(r_0)} > \frac{z'(r_0)}{z(r_0)}r_0.$$

On the other hand, by a straightforward computation, we have

$$\frac{z'(r)}{z(r)} = \frac{2r}{r^2-8} - \frac{2r}{r^2+8} = \frac{32r}{r^4-64},$$

and then

$$\frac{z'(r)r}{z(r)} = \frac{32r^2}{r^4-64} \text{ is decreasing for } r > \sqrt{8}.$$

We use this fact together with (3.34) to obtain

$$(3.35) \quad \frac{\psi^{*'}(R_0)R_0}{\psi^*(R_0)} > \frac{z'(r_0)r_0}{z(r_0)} \geq \frac{z'(R_0)R_0}{z(R_0)},$$

which in turn implies

$$\psi^{*'}(R_0)z(R_0) > z'(R_0)\psi^*(R_0).$$

This inequality contradicts the fact that, by using the second equation in (3.32), we have

$$0 > -R_0 \left(\frac{\psi^*(R_0)}{z(R_0)} \right)' z^2(R_0) = (1-k) \int_{R_0}^{\infty} e^{U(s)} \tilde{\psi}(s) z(s) s ds \geq 0.$$

Thus $\xi_0 \leq \sqrt{8}$ and since the reversed inequality holds as well, then $\xi_0 = \sqrt{8}$. As in Case 1 we conclude that $k = 1$ and then repeat the argument starting with (3.33) to conclude that indeed the second inequality in (3.35) must be an equality. Hence $r_0 = R_0$ and (3.30) shows that $\rho = 8\pi$. In particular, as in Case 1, all the inequalities used in the rearrangement argument are equalities. Therefore it follows once more from Theorem 2.1 that both $\tilde{\Omega}^+$ and $\tilde{\Omega}^-$ are simply-connected and from the Cauchy-Schwarz inequality that $|\nabla\varphi(x)| = |\nabla\varphi(y)|$ whenever $\varphi(x) = \varphi(y) = c$. Since Ω is not simply-connected, there exists a point $P_0 \in \partial\Omega$ such that $\varphi(P_0) = c$ and $\nabla\varphi(P_0) = 0$ which in turn yields $\nabla\varphi(x) = 0$ for $x \in \partial\Omega$. This fact clearly contradicts the strong maximum principle and therefore concludes the proof of Theorem 3.1. \square

4. EXISTENCE OF A SOLUTION AT $\rho = 8\pi$

Solutions of (2.1) are critical points of the functional $I_\rho : H_0^1(\Omega) \mapsto \mathbb{R}$ defined as

$$(4.1) \quad I_\rho(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \rho \log \int_{\Omega} h(x) e^u.$$

A well known consequence of the Moser-Trudinger [38] inequality is that a minimizer u_ρ of I_ρ exists at least in case $\rho < 8\pi$. Actually any such minimizer for $\rho < 8\pi$ is the unique solution to (2.1) according to the uniqueness theorem. In particular, by using the implicit function theorem and the invertibility of the linearized equation at $\rho = 8\pi$ we have (see for example Proposition 6.1 in [13] for a proof):

Lemma 4.1. *The following facts are equivalent:*

- (i) u_ρ converges in $C^2(\overline{\Omega})$ as $\rho \nearrow 8\pi$,
- (ii) A subsequence u_{ρ_n} converges in $C^2(\overline{\Omega})$ as $\rho_n \nearrow 8\pi$,
- (iii) Equation (2.1) possesses a solution at $\rho = 8\pi$.
- (iv) $I_{8\pi}$ attains its infimum.

Let $\tilde{G}(x, p)$ and $\gamma(p)$ be defined in (1.2). We consider the situation where the (unique) branch of minimizers contains a sequence of blowing up solutions, say u_{ρ_n} , as $\rho_n \nearrow 8\pi$. Let q be a blowup point of $u_{\rho_n}(x)$, i.e., there exists a subsequence $u_k \equiv u_{\rho_{n_k}}$ such that

$$u_k(q_k) = \max_{\Omega} u_{\rho_k}(x) \rightarrow +\infty \text{ and } q_k \rightarrow q.$$

Remark 4.2. *In this situation it can be shown that q is a maximum point of $\log h(x) + 4\pi\gamma(x)$ and in particular that*

$$(4.2) \quad \frac{1}{8\pi} \inf_{u \in H_0^1(\Omega)} I_\rho(u) = -1 - \log(\pi) - \sup_{\overline{\Omega}} (h(x) + 4\pi\gamma(x)).$$

These facts are well known. See for example Theorem 1.2 and Lemma 2.3 in [7] for a proof. Actually, q is the unique blow up point of u_ρ as $\rho \nearrow 8\pi$ as shown in the following Proposition 4.1.

Proposition 4.1. *Suppose that a sequence of blowing up solutions, denoted by u_{ρ_n} , exists for (2.1) as $\rho_n \nearrow 8\pi$ and let q be a blow up point. Then, $u_\rho(x)$ converges to $8\pi G(x, q)$ in $C_{loc}^2(\overline{\Omega} \setminus q)$ as $\rho \nearrow 8\pi$.*

Proof. Suppose $q' \neq q$ is a blow up point of $u_{\rho'_n}$. Without loss of generality, we may assume $\rho'_{n-1} < \rho_n < \rho'_n$. Let $\delta > 0$ such that $\delta < \frac{1}{2} \text{dist}(q, q')$ and

$$m_\rho = \frac{\sup_{B(q, \delta)} u_\rho(x)}{\sup_{\overline{\Omega}} u_\rho(x)}.$$

Obviously, m_ρ continuously depends on ρ .

Since u_{ρ_n} and $u_{\rho'_n}$ blows up at q and q' respectively, we have for large n ,

$$m_{\rho_n} = 1 \quad \text{and} \quad m_{\rho'_n} = o(1).$$

Thus, there exists $\rho''_n \in [\rho_n, \rho'_n]$ such that $m_{\rho''_n} = \frac{1}{2}$. Obviously, as $n \rightarrow +\infty$, $\rho''_n \rightarrow 8\pi$ and $\sup_{\overline{\Omega}} u_{\rho''_n} \rightarrow +\infty$. By our choice $m_{\rho''_n} = \frac{1}{2}$, $u_{\rho''_n}$ has at least two blowup points, which is impossible in view of by now standard concentration-compactness results [8], [25] for Liouville-type equations.

Therefore we conclude that any blow up sequence extracted from u_ρ as $\rho \nearrow 8\pi$ admits q as its unique blow up point. It is well known that blow up points are necessarily interior points (see Lemma 2.1 in [13]). Thus the results in [25] apply and we conclude that any such sequence must in fact converge to $8\pi G(x, q)$ in $C_{loc}^2(\overline{\Omega} \setminus q)$ and we conclude in particular, in view of the equivalence of Lemma 4.1 above, that the full branch of minimizers u_ρ satisfies to the same property as well. \square

The following asymptotic estimates for $\rho_n - 8\pi$ along a blowing up sequence u_{ρ_n} was obtained in [13] and [17]:

(j) If $\Delta \log h(q) \neq 0$, then

$$(4.3) \quad \rho_n - 8\pi = c(\Delta \log h(q) + o(1))\lambda_n e^{-\lambda_n}$$

(jj) If $\Delta \log h(q) = 0$, then

$$(4.4) \quad \rho_n - 8\pi = h(q)(D_h(q) + o(1))e^{-\lambda_n},$$

where $\lambda_n = \max_{\overline{\Omega}} u_n - \log(\int_{\Omega} h(x)e^{u_n} dx)$, $o(1) \rightarrow 0$ as $n \rightarrow +\infty$, and

$$(4.5) \quad D_h(q) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B(q, \varepsilon)} \frac{\frac{h(x)}{h(q)} e^{8\pi(\tilde{G}(x, q) - \gamma(q))}}{|x - q|^4} dx - \int_{\Omega^c} \frac{dx}{|x - q|^4}.$$

Remark 4.3. By using the results in either [21] or [23] we see that a sequence of solutions for (2.1) blowing up as $\rho \rightarrow 8\pi$ can be constructed whenever q is a nondegenerate critical point of $\log h(x) + 4\pi\gamma(x)$. Alternatively, by using the condition $D_h(q) \neq 0$, a sequence of solutions for (2.1) blowing up as $\rho \rightarrow 8\pi$ can be constructed by arguing as in [18]. In any case, if q is a blow up point, then $h(q) \neq 0$ by known blow up arguments, see [26].

Theorem 4.1. Let Ω be a C^1 bounded domain and $h(x)$ satisfy (2.2). Then equation (2.1) at $\rho = 8\pi$ admits a solution if and only if there exists a maximum point q of $\log h(x) + 4\pi\gamma(x)$ such that $D_h(q) > 0$.

Proof. We first prove that the condition is sufficient. Suppose $D_h(q) > 0$ for a maximum point q which we can assume without loss of generality to coincide with the origin $q = 0$.

We argue by contradiction and suppose that no solutions exist for $\rho = 8\pi$. In view of Lemma 4.1 we see that necessarily a sequence of blowing up solutions can be found as $\rho_n \nearrow 8\pi$. As observed above, in this situation (4.2) holds. We can assume without loss of generality that $B_1 \subset\subset \Omega$ and then define

$$v_\varepsilon(x) = \begin{cases} 4 \log \frac{1}{|x|} + 8\pi \tilde{G}(x, 0) & \text{for } |x| \geq 1, \\ 2 \log \left(\frac{\varepsilon^2 + 1}{\varepsilon^2 + |x|^2} \right) + 8\pi \tilde{G}(x, 0) & \text{for } |x| \leq 1, \end{cases}$$

so that in particular $v_\varepsilon \equiv 8\pi G(x, q) = 8\pi G(x, 0)$ in $\Omega \setminus B_1$. Then we obtain the following

Lemma 4.4. It holds,

$$I_{8\pi}(v_\varepsilon) = -8\pi - 8\pi \log \pi - 8\pi(\log(h(0)) + 4\pi\gamma(0)) - 8\pi \left(\frac{D_h(0)}{\pi} \right) \varepsilon^2 + O(\varepsilon^3).$$

Proof.

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla v_\varepsilon|^2 dx &= \frac{1}{2} \int_{\Omega \setminus B_1} \nabla \left(\log \frac{1}{|x|^4} + 8\pi \tilde{G}(x, 0) \right) \nabla \log \frac{1}{|x|^4} dx \\ &+ 4\pi \left(\int_{\Omega \setminus B_1} \nabla \log \frac{1}{|x|^4} \nabla \tilde{G}(x, 0) dx + 8\pi \int_{\Omega} \nabla \tilde{G}(x, 0) \nabla \tilde{G}(x, 0) dx \right) \\ &+ \frac{1}{2} \int_{B_1} \left| \frac{4|x|}{\varepsilon^2 + |x|^2} \right|^2 dx + 8\pi \int_{B_1} \nabla \left(2 \log \left(\frac{\varepsilon^2 + 1}{\varepsilon^2 + |x|^2} \right) \right) \nabla \tilde{G}(x, 0) \\ &= \frac{-1}{2} \int_{\partial B_1} \left(\log \frac{1}{|x|^4} + 8\pi \tilde{G}(x, 0) \right) \frac{\partial}{\partial \nu} \left(\log \frac{1}{|x|^4} \right) d\sigma \\ &- 4\pi \int_{\partial B_1} \left(\log \frac{1}{|x|^4} \right) \frac{\partial \tilde{G}}{\partial \nu}(x, 0) d\sigma - 8\pi - 8\pi \log(\varepsilon^2) + 16\pi \varepsilon^2 + O(\varepsilon^4) \end{aligned}$$

$$+8\pi \int_{\partial B_1} \left(2 \log \left(\frac{\varepsilon^2 + 1}{\varepsilon^2 + |x|^2} \right) \right) \frac{\partial \tilde{G}}{\partial \nu}(x, 0) d\sigma$$

$$= -16\pi \log(\varepsilon) - 8\pi + 32\pi^2 \gamma(0) + 16\pi \varepsilon^2 + O(\varepsilon^4),$$

where ν denotes the exterior unit normal. We used here the fact that, since $\tilde{G}(x, 0)$ is harmonic, then

$$\frac{1}{2\pi} \int_{\partial B_1} \tilde{G}(x, 0) d\sigma = \tilde{G}(0, 0) \equiv \gamma(0),$$

and in particular

$$\int_{\partial B_1} \frac{\partial \tilde{G}}{\partial \nu}(x, 0) d\sigma \equiv \int_{\partial \Omega} \frac{\partial \tilde{G}}{\partial \nu}(x, 0) d\sigma = 0.$$

Observe moreover that, since $q = 0$ is a critical point of $\log(h) + 8\pi \tilde{G}$, we have

$$(4.6) \quad h(x) e^{8\pi \tilde{G}(x, 0)} = h(0) e^{8\pi \gamma(0)} + \sum_{i,j=1}^2 b_{ij} x_i x_j + O(|x|^3),$$

where $b_{11} + b_{22} = 0$ because $\log(h) + 8\pi \tilde{G}$ is harmonic in Ω . By using (4.6) and some straightforward evaluation, we can conclude that

$$\begin{aligned} & \int_{B_1 \setminus B_\varepsilon} h(x) e^{8\pi \tilde{G}(x, 0)} \left[\frac{(\varepsilon^2 + 1)^2}{(\varepsilon^2 + |x|^2)^2} - \frac{1}{|x|^4} \right] dx = \\ & -h(0) e^{8\pi \gamma(0)} \frac{\pi(\varepsilon^2 - 1)^2}{2\varepsilon^2} + O(\varepsilon) = h(0) e^{8\pi \gamma(0)} \left[-\frac{\pi}{2\varepsilon^2} + \pi + O(\varepsilon) \right], \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega \setminus B_\varepsilon} \frac{h(x) e^{8\pi \tilde{G}(x, 0)}}{|x|^4} dx = h(0) e^{8\pi \gamma(0)} \int_{\Omega \setminus B_\varepsilon} \frac{\frac{h(x)}{h(0)} e^{8\pi(\tilde{G}(x, 0) - \gamma(0))} - 1}{|x|^4} dx \\ & + h(0) e^{8\pi \gamma(0)} \int_{\mathbb{R}^2 \setminus B_\varepsilon} \frac{dx}{|x|^4} - h(0) e^{8\pi \gamma(0)} \int_{\mathbb{R}^2 \setminus \Omega} \frac{dx}{|x|^4} \\ & = h(0) e^{8\pi \gamma(0)} \left[D_h(0) + \frac{\pi}{\varepsilon^2} + O(\varepsilon) \right], \end{aligned}$$

and

$$\begin{aligned} & \int_{B_\varepsilon} h(x) e^{8\pi \tilde{G}(x, 0)} \frac{(\varepsilon^2 + 1)^2}{(\varepsilon^2 + |x|^2)^2} dx = h(0) e^{8\pi \gamma(0)} \frac{\pi(\varepsilon^2 + 1)^2}{2\varepsilon^2} + O(\varepsilon) \\ & = h(0) e^{8\pi \gamma(0)} \left[\frac{\pi}{2\varepsilon^2} + \pi + O(\varepsilon) \right]. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \int_{\Omega} h(x) e^{u_\varepsilon(x)} dx = \int_{\Omega \setminus B_1} \frac{h(x) e^{8\pi \tilde{G}(x, 0)}}{|x|^4} dx + \int_{B_1} h(x) e^{8\pi \tilde{G}(x, 0)} \frac{(\varepsilon^2 + 1)^2}{(\varepsilon^2 + |x|^2)^2} dx \\ & = \int_{\Omega \setminus B_\varepsilon} \frac{h(x) e^{8\pi \tilde{G}(x, 0)}}{|x|^4} dx - \int_{B_1 \setminus B_\varepsilon} \frac{h(x) e^{8\pi \tilde{G}(x, 0)}}{|x|^4} dx \\ & + \int_{B_1 \setminus B_\varepsilon} h(x) e^{8\pi \tilde{G}(x, 0)} \frac{(\varepsilon^2 + 1)^2}{(\varepsilon^2 + |x|^2)^2} dx + \int_{B_\varepsilon} h(x) e^{8\pi \tilde{G}(x, 0)} \frac{(\varepsilon^2 + 1)^2}{(\varepsilon^2 + |x|^2)^2} dx \end{aligned}$$

$$\begin{aligned}
&= h(0)e^{8\pi\gamma(0)} \left[D_h(0) + \frac{\pi}{\varepsilon^2} - \frac{\pi}{2\varepsilon^2} + \pi + \frac{\pi}{2\varepsilon^2} + \pi + O(\varepsilon) \right] \\
&= h(0)e^{8\pi\gamma(0)} \frac{\pi}{\varepsilon^2} \left[1 + \frac{D_h(0) + 2\pi}{\pi} \varepsilon^2 + O(\varepsilon^3) \right].
\end{aligned}$$

At this point we may collect together the above estimates to conclude that

$$\begin{aligned}
I_{8\pi}(v_\varepsilon) &= \frac{1}{2} \int_{\Omega} |\nabla v_\varepsilon|^2 - 8\pi \log \int_{\Omega} h(x) e^{v_\varepsilon(x)} \\
&= -16\pi \log(\varepsilon) - 8\pi + 32\pi^2\gamma(0) + 16\pi\varepsilon^2 + O(\varepsilon^4) \\
&\quad - 8\pi \log \left(\frac{\pi}{\varepsilon^2} h(0) e^{8\pi\gamma(0)} \right) - 8\pi \log \left(1 + \frac{D_h(0) + 2\pi}{\pi} \varepsilon^2 + O(\varepsilon^3) \right) \\
&= -8\pi - 8\pi \log(\pi) - 8\pi (\log(h(0)) + 4\pi\gamma(0)) + 16\pi\varepsilon^2 - 8\pi \frac{D_h(0) + 2\pi}{\pi} \varepsilon^2 + O(\varepsilon^3) \\
&\quad - 8\pi - 8\pi \log(\pi) - 8\pi (\log(h(0)) + 4\pi\gamma(0)) - 8\pi \frac{D_h(0)}{\pi} \varepsilon^2 + O(\varepsilon^3).
\end{aligned}$$

□

The expansion provided by Lemma 4.4 can be used together with the assumption $D_h(0) > 0$ to obtain

$$\begin{aligned}
\inf_{u \in H_0^1(\Omega)} I_{8\pi}(u) &< -8\pi - 8\pi \log(\pi) - 8\pi (\log h(0) + 4\pi\gamma(0)) \\
&= -8\pi - 8\pi \log(\pi) - 8\pi \sup_{\overline{\Omega}} (\log h(x) + 4\pi\gamma(x)),
\end{aligned}$$

which is in contradiction with (4.2). Hence a solution exists and the sufficiency of the condition is proved.

Next, let us prove the necessary part and suppose that (2.1) at $\rho = 8\pi$ admits a solution. We want to prove in this situation a stronger result, that is, $D_h(q) > 0$ for any maximum point q of $\log h(x) + 4\pi\gamma(x)$. By contradiction we assume that a maximum point q_0 exists such that $D_h(q_0) \leq 0$. The following Lemma 4.5 shows in this case that u_ρ blows up as $\rho \nearrow 8\pi$. This is of course in contradiction with Lemma 4.1 and we may conclude that indeed $D_h(q) > 0$ for all maximum points. □

Lemma 4.5. *Suppose that q is a critical point of $\log h(x) + 4\pi\gamma(x)$ and $D_h(q) \leq 0$. Then there exists a sequence of solutions u_n of (2.1) with $\rho_n < 8\pi$ and $\rho_n \nearrow 8\pi$ such that u_n blows up at q .*

Proof. The proof will be divided in two cases.

Case 1: $D_h(q) < 0$.

If q is a nondegenerate critical point of $\log h(x) + 4\pi\gamma(x)$, then by Remark 4.3 we can construct a sequence of blowing-up solutions u_n of (2.1) with $\rho = \rho_n$ whose unique blow up point is q (see Proposition 4.1). Of course $\rho_n < 8\pi$ because of (4.4) and $D_h(q) < 0$.

If q is a degenerate critical point we assume without loss of generality that

$$\frac{\partial^2}{\partial x_1 \partial x_2} (\log h(x) + 4\pi\gamma(x)) = 0 \quad \text{at } x = q.$$

Then we let

$$h_\varepsilon(x) = h(x) \exp \left(\varepsilon [(x_1 - q_1)^2 - (x_2 - q_2)^2] \right),$$

where $q = (q_1, q_2)$. It is easy to see that $h_\varepsilon(x)$ satisfies (2.2), and q is a nondegenerate critical point of $\log h_\varepsilon(x) + 4\pi\gamma$ for any $\varepsilon > 0$. Let $D_\varepsilon(q)$ be the quantity defined in (4.5), where $h(x)$ is replaced by $h_\varepsilon(x)$. Obviously for small $\varepsilon > 0$ we have $D_\varepsilon(q) < 0$.

Let $(2.1)_\varepsilon$ denote problem (2.1) where $h(x)$ has been replaced by $h_\varepsilon(x)$. Since $h_\varepsilon(x)$ satisfies (2.2), then $(2.1)_\varepsilon$ admits a unique solution $u_\rho^\varepsilon(x)$ for any $\rho < 8\pi$. Since q is a nondegenerate critical point of $\log h_\varepsilon + 4\pi\gamma$ then we can construct a sequence of solutions for $(2.1)_\varepsilon$ which blows up at q (see Remark 4.3). Thus, by using (4.4), the fact that $D_\varepsilon(q) < 0$ and the uniqueness theorem, we conclude that $u_\rho^\varepsilon(x)$ coincides with this sequence and hence blows up as $\rho \nearrow 8\pi$.

Let C be a fixed large positive number and δ be a small positive number. Then for each $\varepsilon > 0$, there exists a $\rho^\varepsilon \in (0, 8\pi)$ such that the solution $u_\varepsilon = u_{\rho^\varepsilon}^\varepsilon$ of $(2.1)_\varepsilon$ with $\rho = \rho^\varepsilon$ satisfies

$$(4.7) \quad \max_{\bar{\Omega}} u_\varepsilon(x) = C \quad \text{and} \quad \sup_{B(q, \delta)} u_\varepsilon(x) = C \quad \geq \quad 2 \sup_{\bar{\Omega} \setminus B(p, \delta)} u_\varepsilon(x).$$

By letting $\varepsilon \rightarrow 0$, and $\rho^\varepsilon \rightarrow \rho(C) \in (0, 8\pi]$, there exists a solution $u(x; C)$ of (2.1) such that (4.7) holds. Clearly Theorem 3.1 implies that $u(x; C_1) \neq u(x; C_2)$ and $\rho(C_1) \neq \rho(C_2) < 8\pi$ whenever $C_1 \neq C_2$. As $C \rightarrow +\infty$ we obtain a sequence of solutions for (2.1) which blows up at q which is the desired conclusion in **Case 1**.

Case 2: $D_h(q) = 0$

For $0 < t < 1$ we define

$$d(t) = \int_{\Omega} \frac{\left(\frac{h(x)}{h(q)} \right)^t e^{8\pi t(\tilde{G}(x, q) - \gamma(q))} - 1}{|x - q|^4} dx - \int_{\Omega^c} \frac{dx}{|x - q|^4}.$$

Clearly,

$$(4.8) \quad d'(t) = \int_{\Omega} \frac{\left[\log \left(\frac{h(x)}{h(q)} \right) + 8\pi(\tilde{G}(x, q) - \gamma(q)) \right] \left(\frac{h(x)}{h(q)} \right)^t e^{8\pi t(\tilde{G}(x, q) - \gamma(q))}}{|x - q|^4} dx$$

and

(4.9)

$$d''(t) = \int_{\Omega} \frac{\left[\log \left(\frac{h(x)}{h(q)} \right) + 8\pi(\tilde{G}(x, q) - \gamma(q)) \right]^2 \left(\frac{h(x)}{h(q)} \right)^t e^{8\pi t(\tilde{G}(x, q) - \gamma(q))}}{|x - q|^4} dx.$$

Since $\log \left(\frac{h(x)}{h(q)} \right) + 8\pi(\tilde{G}(x, q) - \gamma(q))$ is harmonic and $\log \left(\frac{h(x)}{h(q)} \right) + 8\pi(\tilde{G}(x, q) - \gamma(q)) = O(|x - q|^2)$, then the integral defining $d'(t)$ is well-defined in the sense of the following limit

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B(q, \varepsilon)} \frac{\left[\log \left(\frac{h(x)}{h(q)} \right) + 8\pi(\tilde{G}(x, q) - \gamma(q)) \right] \left(\frac{h(x)}{h(q)} \right)^t e^{8\pi t(\tilde{G}(x, q) - \gamma(q))}}{|x - q|^4} dx.$$

Since $d(1) = D_h(q) = 0$ and $d(0) = -\int_{\Omega^c} \frac{dx}{|x - q|^4} < 0$, by using $d''(t) > 0$, we have $d(t) < 0$ for $t \in [0, 1)$, that is

$$(4.10) \quad \int_{\Omega} \frac{\left(\frac{h(x)}{h(q)} \right)^t e^{8\pi t(\tilde{G}(x, q) - \gamma(q))} - 1}{|x - q|^4} dx - \int_{\Omega^c} \frac{dx}{|x - q|^4} < 0.$$

Let $h_{\epsilon}(x) = (h(x))^{1-\epsilon} e^{-8\pi\epsilon\tilde{G}(x, q)}$. Then q is a critical point of $\log h_{\epsilon}(x) + 8\pi\tilde{G}(x, q)$, and

$$D_{\epsilon}(q) = \int_{\Omega} \frac{\frac{h_{\epsilon}(x)}{h_{\epsilon}(q)} e^{8\pi(\tilde{G}(x, q) - \gamma(q))} - 1}{|x - q|^4} dx - \int_{\Omega^c} \frac{dx}{|x - q|^4} = d(1 - \epsilon) < 0.$$

Now we consider

$$(4.11) \quad \begin{cases} \Delta u^{\epsilon} + \rho \frac{h_{\epsilon}(x) e^{u^{\epsilon}}}{\int_{\Omega} h_{\epsilon}(x) e^{u^{\epsilon}} dx} & \text{in } \Omega \\ u^{\epsilon} = 0 & \text{on } \partial\Omega \end{cases}$$

Note that

$$\begin{aligned} \log h_{\epsilon}(x) &= (1 - \epsilon) \log(h(x)) - 8\pi\epsilon\tilde{G}(x, q) \\ &= (1 - \epsilon) \log h(x) - 8\pi\epsilon G(x, q) - 4\epsilon \log |x - q|. \end{aligned}$$

By (2.2) $\log h(x)$ can be extended to Ω^* as a subharmonic function. As above, $G(x, q)$ can also be extended to Ω^* by setting

$$\widehat{G}(x, q) = \begin{cases} G(x, q) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega^* \setminus \Omega. \end{cases}$$

Thus, we can extend $\log h_{\epsilon}(x)$ to the larger domain Ω^* . However, the extended function $-\widehat{G}(x, q)$ is not subharmonic in Ω^* . Hence $\log h_{\epsilon}(x)$ does not satisfy (2.2) and therefore Theorem 3.1 cannot be applied to (4.11). Nevertheless, we will see in the following that the uniqueness theorem is still valid for (4.11) provided that ϵ is small enough.

Since for $\rho = 0$ the linearized problem has positive first eigenvalue, then, of course, there is a small $\rho_0 > 0$, which do not depend on ϵ , such that there is only one solution to (4.11) for $\rho \leq \rho_0$. For $\rho > \rho_0$ we set

$$u_\epsilon^*(x) = \begin{cases} u^\epsilon(x) - 8\pi\epsilon[G(x, p) + \frac{1}{2\pi} \log |x - p|], & \text{if } x \in \Omega \\ -4\epsilon \log |x - p|, & \text{if } x \in \Omega^* \setminus \Omega, \end{cases}$$

where $u^\epsilon(x)$ is a solution for (4.11). Then $u_\epsilon^*(x) \in C(\Omega^*)$ and

$$(4.12) \quad \Delta u_\epsilon^*(x) + \rho \frac{h^{1-\epsilon}(x) e^{u_\epsilon^*(x)}}{\int_\Omega h^{1-\epsilon} e^{u_\epsilon^*(x)} dx} \geq 0 \quad \text{in } \Omega^*,$$

in the distribution sense, provided that the following holds:

$$(4.13) \quad \frac{\partial}{\partial \nu} [u^\epsilon(x) - 8\pi\epsilon G(x, q)] \leq 0 \quad \text{for } x \in \Omega^* \cap \partial\Omega.$$

We prove (4.13) by contradiction. Suppose that there exists a sequence of solutions $u_{\epsilon_k}^{\rho_k}$ for (4.11) \equiv (4.11) $_{\epsilon_k}$ with $\rho_k \geq \rho_0$ such that

$$(4.14) \quad \frac{\partial u_{\epsilon_k}^{\rho_k}(x_k)}{\partial \nu} \geq -C\epsilon_k \quad \text{for some } x_k \in \partial\Omega,$$

where $-C = \inf_{x \in \partial\Omega} \frac{\partial G(x, q)}{\partial \nu}$.

If $u_{\epsilon_k}^{\rho_k}$ is uniformly bounded in $\overline{\Omega}$, then there is a subsequence of $u_{\epsilon_k}^{\rho_k}$, which converges to a function u which satisfies

$$\begin{cases} \Delta u + \rho_0^* \frac{h_{\epsilon_0^*}(x) e^{u(x)}}{\int_\Omega h_{\epsilon_0^*}(x) e^{u(x)} dx} = 0 & \text{in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \text{ and } \frac{\partial u}{\partial \nu}(x_0) \geq 0 & \text{for some } x_0 \in \partial\Omega, \end{cases}$$

where $\epsilon_0^* = \lim_{k \rightarrow +\infty} \epsilon_k$, $\rho_0^* = \lim_{k \rightarrow +\infty} \rho_k \geq \rho_0 > 0$ and $x_0 = \lim_{k \rightarrow +\infty} x_k$. Since $u > 0$ on $\partial\Omega$, then the Hopf boundary Lemma says that $\frac{\partial u}{\partial \nu}(x) < 0$ for all $x \in \partial\Omega$, which is a contradiction to $\frac{\partial u}{\partial \nu}(x_0) \geq 0$.

On the other side, if there exists a blowing up subsequence (which we denote by u_k) of $u_{\epsilon_k}^{\rho_k}$, then $\rho_k \rightarrow 8\pi$, $\epsilon_k \rightarrow \epsilon_0^*$, and $\rho_k \frac{h_{\epsilon_k}(x) e^{u_k}}{\int_\Omega h_{\epsilon_k}(x) e^{u_k} dx} \rightarrow 8\pi \delta_{x_1}$ for some $x_1 \in \Omega$. Furthermore, $u_k \rightarrow G(x, x_1)$ in $C^2(\overline{\Omega} \setminus \{x_1\})$. At this point (4.14) implies $\frac{\partial G(x_0, x_1)}{\partial \nu} = \lim_{k \rightarrow +\infty} \frac{\partial u_k}{\partial \nu}(x_k) \geq 0$, where $x_0 = \lim_{k \rightarrow +\infty} x_k$, which is once more a contradiction to the Hopf boundary Lemma. Hence (4.13) holds for any ϵ small enough.

Since u_ϵ^* satisfies the differential inequality (4.12), we can follow the proof of Theorem 2.1 to show that the Bol's inequality holds for u^ϵ , i.e. for any $\omega \Subset \Omega$, we have

$$2\ell_\epsilon^2(\partial\omega) \geq m_\epsilon(\omega)(8\pi - m_\epsilon(\omega)),$$

$$\text{where } \ell_\epsilon(\partial\omega) = \int_{\partial\omega} e^{v_\epsilon(x)/2} ds, \quad m_\epsilon(\partial\omega) = \int_\omega e^{v_\epsilon(x)} ds,$$

and

$$v_\epsilon(x) = u^\epsilon(x) + \log h_\epsilon(x) + \log \rho - \log \int_{\Omega} h_\epsilon(x) e^{u^\epsilon(x)} dx.$$

By using the Bol's inequality, we can follow the proof of Theorem 3.1 to show that equation (4.11) admits at most one solution for $\rho \leq 8\pi$ as well for any ϵ small enough.

Since $D_\epsilon(q) < 0$, then we can apply the result obtained in **Case 1**. Therefore the solution $u_\epsilon^\rho(x)$ of (4.11) blows up at q as $\rho \nearrow 8\pi$. However, the same argument adopted in **Case 1** shows that $u^\rho(x)$, the solution of (2.1), blows up at q as well, which is the desired result. \square

Remark 4.6. *The proof of Theorem 4.1 shows in particular that if there exists a maximum point q of $\log h(x) + 4\pi\gamma(x)$ with $D_h(q) > 0$, then we have $D_h(p) > 0$ for any other maximum point p . See Lemma 4.5 and the few lines above it.*

The following results will provide us with a proof of Corollaries 1.2 and 1.3 in the more general situation where (2.1) is concerned. The situation where Ω is simply-connected has been already discussed in [13] and we will not pursue it here any further. The nondegeneracy of the maximum point as stated in Corollary 1.2 requires a more subtle analysis which is the content of Theorem 4.2 below.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 . If q is a critical point of $\log h(x) + 4\pi\gamma(x)$ with $D_h(q) \leq 0$, then q is a maximum point. Furthermore, q is the unique maximum point.*

Proof. By Lemma 4.5 and Proposition 4.1 we see that q is the blow up point of a sequence of blowing up solutions. Therefore q is a maximum point, see Remark 4.2. To prove the uniqueness of any such maximum point, observe that Remark 4.2 says that indeed $D_h(p) \leq 0$ for any other maximum point. Now suppose that there exists another maximum point $q' \neq q$. Then Lemma 4.5 yields a sequence of blowing up solutions $u_{\rho'_n}$ whose blow up point should be q' . Of course, this is a contradiction to Proposition 4.1. \square

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain of class C^1 . If $\log h(x) + 4\pi\gamma(x)$ admits more than one maximum point, then equation (2.1) has a solution for $\rho = 8\pi$.*

Proof. Let $q_1 \neq q_2$ be maximum points of $\log h(x) + 4\pi\gamma(x)$. We deduce from Corollary 4.1 that $D_h(q_1) > 0$ and $D_h(q_2) > 0$. Hence Theorem 4.1 yields the existence of a solution for (2.1). \square

Now we are in the position to prove:

Theorem 4.2. *Let q be a critical point of $\log h(x) + 4\pi\gamma(x)$ with $D_h(q) \leq 0$. Then q is a nondegenerate critical point.*

Proof. Since $D_h(q) \leq 0$ we deduce from Corollary 4.1 that q is the unique maximum point. We argue by contradiction and suppose that q is degenerate. Without loss of generality, we may assume that $q = 0$ and

$$(4.15) \quad \frac{\partial^2}{\partial x_1^2}(\log h + 4\pi\gamma) \Big|_{x=0} = \frac{\partial^2}{\partial x_1 \partial x_2}(\log h + 4\pi\gamma) \Big|_{x=0} = 0,$$

$$(4.16) \quad \frac{\partial^2}{\partial x_2^2}(\log h + 4\pi\gamma) \Big|_{x=0} = a \leq 0.$$

The proof will be divided in two cases.

Case 1: $D_h(q) < 0$.

We set

$$(4.17) \quad \log h_\varepsilon(x) = \log h(x) + \varepsilon(x_1^2 - x_2^2),$$

and let $D_\varepsilon(q)$ be defined by (4.5) where h has just been replaced by h_ε . Then $D_\varepsilon(q) < 0$ if ε is sufficiently small. Since $x_1^2 - x_2^2$ is harmonic, then Corollary 4.1 can be applied and we conclude that $q = 0$ should be a maximum point. This is impossible as one readily verifies by using (4.15) and (4.16) together with (4.17). Hence the desired conclusion in Case 1 is established.

Case 2: $D_h(q) = 0$.

We set

$$h_t(x) = h(x)e^{-tx_1^4}, \quad t > 0.$$

Thus $q = 0$ is a critical point of $h_t(x)$ and

$$D_{h,t}(0) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B(0;\varepsilon)} \frac{h(x)}{h(0)} \frac{e^{-tx_1^4} e^{8\pi(\tilde{G}(x,0) - \gamma(0))} - 1}{|x|^4} - \int_{\Omega^c} \frac{dx}{|x|^4}.$$

Clearly, $D_{h,t}(0) < D_h(0)$ for $t > 0$. Next we consider the mean field equation:

$$(4.17)_t \quad \begin{cases} \Delta u^t + \rho \frac{h(x)e^{-tx_1^4} e^{ut}}{\int_{\Omega} h(x)e^{-tx_1^4} e^{ut} dx} = 0 & \text{in } \Omega, \\ u^t = 0 & \text{on } \partial\Omega. \end{cases}$$

Although h_t is not subharmonic, we can argue as in Case 2 of the proof of Lemma 4.5 to show that $(4.17)_t$ admits at most one solution for $\rho \in [0, 8\pi]$ and for any t small enough. For a fixed small $t > 0$, $D_{h,t}(0) < 0$ and then the conclusion obtained in Case 1 above says that $q = 0$ is a nondegenerate critical point of $\log h(x) + 4\pi\gamma(x) - tx_1^4$. Hence, it should be a nondegenerate critical point of $\log h(x) + 4\pi\gamma(x)$ as well, which is the desired contradiction in Case 2. \square

Finally we have the following generalized version of Corollary 1.4

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and multiply-connected domain of class C^1 . Then*

$$\frac{1}{8\pi} \inf_{u \in H_0^1(\Omega)} I_\rho(u) \leq -1 - \log(\pi) - \sup_{x \in \overline{\Omega}} (\log h(x) + 4\pi\gamma(x)),$$

and (2.1) admits a solution at $\rho = 8\pi$ if and only if the strict inequality holds.

Proof. Theorem 4.1 says that a solution at $\rho = 8\pi$ exists if and only if $D_h(q) > 0$ for a maximum point of $\log h(x) + 4\pi\gamma(x)$. Hence Lemma 4.4 shows immediately that if a solution exists, then the inequality is strict. On the other side, if we assume by contradiction that the inequality is strict but no solution exists at $\rho = 8\pi$, then we get a contradiction to (4.2). \square

5. EQUIVALENCE OF STATISTICAL ENSEMBLES.

Our main concern in this section is the applicability of Theorem 1.2 to some long standing open problems in the statistical mechanics analysis of two dimensional turbulence [11]. This is why in some statements we will assume the domain's regularity taken up in Theorem 1.2, see Theorems 5.3 and 5.4 below. Let

$$s(t) = \begin{cases} -t \log t, & t > 0 \\ 0, & t = 0, \end{cases}$$

and $\Omega \subset \mathbb{R}^2$ be a bounded domain. We define

$$\mathcal{P}_\Omega = \left\{ \rho \in L^1(\Omega) \mid \rho \geq 0 \text{ a.e. in } \Omega, \int_\Omega \rho = 1, \int_\Omega (-s(\rho)) < +\infty \right\},$$

and let $G(x, y)$ be the Green's function on Ω as defined in the introduction. For any $\rho \in \mathcal{P}_\Omega$ let us set

$$\mathcal{S}(\rho) = \int_\Omega s(\rho), \quad \mathcal{E}(\rho) = \frac{1}{2} \int_\Omega \rho G[\rho],$$

where

$$G[\rho](x) = \int_\Omega G(x, y) \rho(y) dy,$$

and

$$\mathcal{F}_\beta(\rho) = -\frac{1}{\beta} \mathcal{S}(\rho) + \mathcal{E}(\rho).$$

For any $E \in \mathbb{R}$ we consider the Microcanonical Variational Principle (MVP for short)

$$(5.18) \quad S(E) = \sup\{\mathcal{S}(\rho), \rho \in \mathcal{P}_\Omega(E)\}, \quad \mathcal{P}_\Omega(E) = \{\rho \in \mathcal{P}_\Omega \mid \mathcal{E}(\rho) = E\},$$

while for any $-8\pi \leq \beta < 0$ we consider the Canonical Variational Principle (CVP for short)

$$(5.19) \quad f(\beta; \Omega) = \sup\{\mathcal{F}_\beta(\rho), \rho \in \mathcal{P}_\Omega\}.$$

For each $0 < \lambda \leq 8\pi$ let us set

$$g_\lambda(\Omega) := \sup_{u \in H_0^1(\Omega)} \mathcal{J}_\lambda(u), \quad \mathcal{J}_\lambda(u) = -\frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{\lambda} \log \left(\int_\Omega e^{\lambda u} \right).$$

The following results are well known. Although not essentials for the incoming discussion, they are quite relevant to understand how $f(\beta; \Omega)$ (the

physical free energy) is related with $g_\lambda(\Omega)$ (which is essentially our functional J_ρ). Because of this basic role, we will provide a sketchy proof for the sake of completeness. However we will not discuss the case where $\beta > 0$ (that is $\lambda < 0$) which is much easier, see [11].

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain satisfying a uniform cone property [1]. For any $-8\pi < \beta < 0$ and for any $0 < \lambda < 8\pi$ the supremums $f(\beta; \Omega)$ and $g_\lambda(\Omega)$ are attained and in particular, setting $\lambda = -\beta$,*

$$f(\beta; \Omega) = g_\lambda(\Omega).$$

Proof. By using Theorem 2.1 in [10] we see that for any $-8\pi < \beta < 0$ the supremum $f(\beta; \Omega)$ is attained by a density $\rho_\beta \in \mathcal{P}$ which solves the MFE (Mean Field Equation)

$$\rho_\beta(x) = \frac{e^{-\beta G[\rho_\beta](x)}}{\int_\Omega e^{-\beta G[\rho_\beta]}}, \quad Q(\beta, \Omega).$$

Actually the argument in [10] relies on the evaluation of the thermodynamic limit for a renormalized free energy functional. It turns out that another proof of this fact which uses variational type arguments based on the logarithmic Hardy-Sobolev inequality [27] can be found in Lemmas 2.1 and 2.2 in [16] under stronger regularity assumptions on Ω .

The Moser-Trudinger inequality [38] and the direct method in the calculus of variations show that for any $0 < \lambda < 8\pi$ the supremum $g_\lambda(\Omega)$ is attained by a function $v = v_\lambda \in H_0^1(\Omega)$ which solves the MFE

$$\begin{cases} -\Delta v = \frac{e^{\lambda v}}{\int_\Omega e^{\lambda v}} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad P(\lambda, \Omega).$$

Let ρ_β be a maximizer for $\beta > -8\pi$, put $\beta = -\lambda$ and $v_\lambda = G[\rho_{-\lambda}]$. Then we see that v_λ is in $H_0^1(\Omega)$ and solves $P(\lambda, \Omega)$. Hence, by a straightforward evaluation, we obtain

$$f(-\lambda; \Omega) = \mathcal{F}_{-\lambda}(\rho_{-\lambda}) = \mathcal{J}_\lambda(v_\lambda).$$

Clearly, the same equality, to be read in the opposite direction, shows that if $v = v_\lambda$ is a maximizer for $\lambda < 8\pi$ and we define $\rho_{-\lambda} = \frac{e^{\lambda v_\lambda}}{\int_\Omega e^{\lambda v_\lambda}}$, then it clearly solves $Q(-\lambda, \Omega)$ and in particular

$$g_\lambda(\Omega) = \mathcal{F}_{-\lambda}(\rho_{-\lambda}).$$

We easily deduce at this point that $g_\lambda(\Omega)$ and $f(-\lambda; \Omega)$ must coincide. \square

Remark 5.1. *The proof above shows in particular that ρ_β solves $Q(\beta, \Omega)$ if and only if, setting $\lambda = -\beta$, then $v_\lambda = G[\rho_{-\lambda}]$ belongs to $H_0^1(\Omega)$ and weakly solves $P(\lambda, \Omega)$. Clearly $P(\lambda, \Omega)$ is equivalent to problem (1.1) as far as $\lambda \neq 0$.*

Remark 5.2. Since \mathcal{S} is concave, if $\rho \in \mathcal{P}_\Omega$, by the Jensen's inequality we have

$$0 = s\left(\int_\Omega \rho\right) \geq \int_\Omega s(\rho) = \mathcal{S}(\rho),$$

that is, $\mathcal{S}(\rho) \leq 0$, $\forall \rho \in \mathcal{P}_\Omega$.

Theorem 5.2. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. For any $E > 0$, $S(E) < +\infty$ and there exists $\rho^{(E)} \in \mathcal{P}_\Omega(E)$ such that $S(E) = \mathcal{S}(\rho^{(E)})$. In particular $S(E)$ is continuous, and letting $E_0 = \mathcal{E}\left(\frac{1}{|\Omega|}\right)$, then $S(E_0) = 0$ and S is strictly increasing for $E < E_0$ and strictly decreasing for $E > E_0$. Moreover for each E there exists $\beta \in \mathbb{R}$ such that $\rho^{(E)}$ solves $Q(\beta, \Omega)$.

Proof. This is the content of Propositions 2.1, 2.2, 2.3 and 2.4 in [11]. \square

Theorem 5.2 shows that the MVP always has a solution (that is, the supremum in (5.18) is always attained) which consequently describes the (mean field) thermodynamic of the system. This is no longer true for the CVP which surely won't have a solution if $\beta < -8\pi$ (that is, the supremum in (5.19) is not attained). The characterization of those cases where these two formulations yield the same thermodynamics (equivalence of microcanonical and canonical statistical ensembles) is one of the main aim in the statistical mechanics description of the system. The following results are concerned with the solution of this problem.

Let us recall that $P(\lambda, \Omega)$ has been introduced during the proof Theorem 5.1 to denote one of the equivalent formulations of the mean field equation, see Remark 5.1.

In [11], a bounded domain is said to be:

- (-) of **first kind**, if solutions of $P(\lambda, \Omega)$ blow up as $\lambda \nearrow 8\pi^-$;
- (-) of **second kind**, otherwise.

Among many other things, it was shown in [11] that if the inequality is strict in Corollary 1.4 then Ω is of second kind. On this basis some examples of simply-connected domains of second kind were exhibited there, but a full characterization was still missing. This problem was solved in [13], where Chang, Chen and the second author characterized domains of the first/second kind in case Ω is simply-connected. More recently this result has been extended in [7] to cover the case where Dirac data are included in (2.1). We complete those results here with a full characterization of domains of first/second kind. In particular the following Theorem has to be complemented with Theorem 1.1 and Corollary 1.4 (and Theorems 1.1 and 1.5 in [13]) which provide other necessary and sufficient conditions for a fixed domain to be of first or second kind. For example, it has been already observed in the introduction that the annulus $B(0, 1) \setminus B(x_0, \epsilon)$ with $\epsilon < 1 - |x_0|$ is of second kind if $x_0 = 0$ while if $x_0 \neq 0$ and ϵ is small enough then it is of first kind.

Theorem 5.3. Let Ω be an open, bounded domain of class C^1 . The following facts are equivalent:

- (-) Ω is of first kind;
 - (-) $g_{8\pi}(\Omega)$ is not attained;
 - (-) $P(8\pi, \Omega)$ has no solution;
 - (-) The unique branch of maximizers for $g_\lambda(\Omega)$, $\lambda < 8\pi$ blows up as $\lambda \nearrow 8\pi$.
- Moreover, the following facts are equivalent:
- (-) Ω is of second kind;
 - (-) $g_{8\pi}(\Omega)$ is attained;
 - (-) $P(8\pi, \Omega)$ admits a solution $u_{8\pi}$;
 - (-) The unique branch of maximizers u_λ for $g_\lambda(\Omega)$, $\lambda < 8\pi$ converges uniformly to $u_{8\pi}$ as $\lambda \nearrow 8\pi$.

Proof. By using Theorem 1.2 and the implicit function Theorem the proof can be worked out as in Proposition 6.1 in [13]. \square

Remark 5.3. We remark that if Ω is simply-connected then Theorem 5.3 holds even if Ω has a finite number of conical-type singular points, see [13] for further details and a discussion of the first/second kind issue for some natural domains such as rectangles and polygons.

By using the uniqueness of solutions for $P(\lambda, \Omega)$ with $\lambda < 8\pi$ obtained in [46], and under certain further assumptions on the topology of Ω , in [11] the authors were able to establish the equivalence of statistical ensembles, namely the above mentioned equivalence of the variational principles (5.18) and (5.19). Indeed, since the uniqueness in [46] was obtained just for simply-connected (smooth and bounded) domains, they restrict their attention to this class. By using Theorem 1.2 we are able to extend those results to the general case of bounded domains of class C^1 . As a matter of fact, the proof adopted in [11] works fine as well, the unique modification being just that of using Theorem 1.2 instead of the Suzuki's [46] uniqueness result. This is why we will not repeat those proofs in full details here. We remark that in [13] and [6] the full uniqueness theory presented here was developed under much weaker smoothness assumptions on Ω . Let

$$E = E(\beta) = \mathcal{E}(\rho_\beta),$$

be the energy of the (unique, see Remark 5.1 and Theorem 1.2) solution of $P(\lambda, \Omega)$ where $\lambda = -\beta$ for $0 < \lambda < 8\pi$. Hence $E : (-8\pi, 0) \mapsto \mathbb{R}^+$ is well defined. As in [11], if Ω is of first kind we set $E_c = E(-8\pi) = +\infty$, while if it is of second kind we set $E_c = E(-8\pi) < +\infty$. Once more, as already mentioned above, we will not discuss the situation where $\beta \geq 0$ which is easier, see [11].

We finally have the generalization of Proposition 3.3 in [11] to the case where Ω is an open, bounded and multiply connected domain of class C^1 .

Theorem 5.4. *Let Ω be an open and bounded domain of class C^1 . We assume that either Ω is of first kind and $E \in (0, +\infty)$ or Ω is of second kind and $E \in (0, E_c)$. Then we have:*

- (i) $F(\beta) = -\beta f(\beta; \Omega)$ is defined for $\beta \geq -8\pi$, strictly convex and decreasing;
- (ii) F is differentiable for $\beta > -8\pi$ and $E(\beta) = -F'(\beta) = \frac{1}{2} \int_\Omega \rho_\beta G[\rho_\beta]$, where ρ_β solves $Q(\beta, \Omega)$. In particular $E(\beta)$ is a continuous and strictly

monotone decreasing bijection;

(iii) $S(E) = \inf_{\beta} \{F(\beta) + \beta E(\beta)\}$ and hence is a smooth and concave function of E ;

(iv) If $\rho^{(E)}$ is a maximizer for (5.18) then $\rho_{\beta} = \rho^{(E(\beta))}$. In particular $\rho^{(E)}$ solves $P(-\beta; \Omega)$ (equivalence of statistical ensembles for $E < E_c$) and the solution is unique.

Proof. Of course, we restrict our attention to the case where Ω is multiply-connected, the other case being already included in [11] Proposition 3.3.

(i) The proof can be worked out as in Proposition 7.3 in [10].

(ii) We argue as in [11] making use of Theorem 1.2 above. Let $-8\pi < \beta_i < 0$, $i = 1, 2$ and ρ_i , $i = 1, 2$ be the corresponding maximizers of (5.19). Clearly

$$E(\beta_i) = \mathcal{E}(\rho_i), \quad i = 1, 2.$$

Hence

$$F(\beta_2) \geq \mathcal{S}(\rho_1) - \beta_2 \mathcal{E}(\rho_1) \equiv F(\beta_1) - (\beta_2 - \beta_1)E(\beta_1),$$

$$F(\beta_1) \geq \mathcal{S}(\rho_2) - \beta_1 \mathcal{E}(\rho_2) \equiv F(\beta_2) - (\beta_1 - \beta_2)E(\beta_2).$$

The last two inequalities imply

$$-E(\beta_2) \leq \frac{F(\beta_2) - F(\beta_1)}{\beta_2 - \beta_1} \leq -E(\beta_1), \quad \text{if } \beta_1 > \beta_2,$$

and

$$-E(\beta_1) \leq \frac{F(\beta_2) - F(\beta_1)}{\beta_2 - \beta_1} \leq -E(\beta_2), \quad \text{if } \beta_2 > \beta_1.$$

Let $v_i := G[\rho_i]$, $i = 1, 2$ be the corresponding solutions of $P(\lambda_i, \Omega)$, $\lambda_i = -\beta_i$, $i = 1, 2$. Then, in view of Remark 5.1 and Theorem 1.2 we see that, as $\beta_1 \rightarrow \beta_2$, then $v_1 := G[\rho_1] \rightarrow v_2 := G[\rho_2]$ in $H_0^1(\Omega)$. In particular it is not difficult to verify that

$$E(\beta_1) = \frac{1}{2} \int_{\Omega} \rho_1 v_1 \rightarrow \frac{1}{2} \int_{\Omega} \rho_2 v_2 = E(\beta_2), \quad \text{as } \beta_1 \rightarrow \beta_2.$$

Therefore we conclude that

$$-F'(\beta) = E(\beta) = \frac{1}{2} \int_{\Omega} \rho_{\beta} G[\rho_{\beta}], \quad \forall \beta \in (-8\pi, 0).$$

The continuity of $E(\beta)$ follows once more from Theorem 1.2. Finally $E = -F'$ is strictly monotone and decreasing (hence a bijection) since F is strictly convex.

(iii)-(iv) In view of (ii), the proof provided in Proposition 3.3(iii)-(iv) of [11] works exactly as it stands. \square

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